## Laplace Transforms

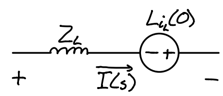
$f(t) \longrightarrow \bar{f}(s)$, where $s = \underset{\text{real part}}{\overset{\text{complex number}}{\sigma}} + \overset{\sqrt{-1}}{j}\,\overset{\text{imaginary part}}{\omega}$

- Complex Number Representation
  - $\alpha + \beta j \longrightarrow$ Binomial Representation
  - $(\alpha, \beta) \longrightarrow$ Cartesian Representation
  - $\sqrt{\alpha^2+\beta^2}\ \angle \tan\left(\frac{\beta}{\alpha}\right) = \sqrt{\alpha^2+\beta^2} \cdot e^{j\tan\left(\frac{\beta}{\alpha}\right)} \longrightarrow$ Polar Representation

- Euler's Identity: $e^{jx} = \cos x + j\sin x$
- Conjugate: $(\alpha + \beta j)^* = \alpha - \beta j$

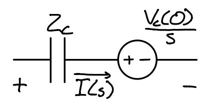
- Inductor Laplace Transform

$$v(t) = L i'(t) \longrightarrow V(s) = Ls\,I(s) - L i_L(0)$$
$$\hookrightarrow Z_L = Ls$$




- Capacitor Laplace Transform

$$\frac{V_c(0)}{s} = \frac{q_0}{Cs}$$
$$v(t) = \frac{1}{C}\int i(t)\,dt \longrightarrow V(s) = (Cs)^{-1} I(s) + (Cs)^{-1}\int_{-\infty}^{0} i_c(t)\,dt$$
$$\hookrightarrow Z_c = (Cs)^{-1}$$




- Solving Method
  - Find I.C.: $i_L(0), V_c(0)$
  - Assume steady state for $t<0$
  - After $t=0$, circuit enters Laplace Domain $\longrightarrow$ apply capacitor/inductor laplace transforms

## Phasors

- A Phasor is a complex number representing a sinusoid
- The function $f(t) = A\cos(\omega t + \alpha)$ can be represented by the phasor:

$$F = \left(\frac{A}{\sqrt{2}}\right)\angle\alpha$$

Peak Value ($V_m / i_m$) — $A$

RMS Value ($V_{rms} / i_{rms}$) — $\frac{A}{\sqrt{2}}$

- Phasor magnitude is RMS value; Sinusoid magnitude is Peak value

- Phasor voltage/current in inductor: $\bar{V}_L = j\omega L\bar{i}$ ; $\bar{i} = i\angle\alpha$ ; $Z_L = j\omega L$
- Phasor voltage/current in capacitor: $\bar{V}_c = \frac{\bar{i}}{j\omega C}$ ; $\bar{i} = i\angle\alpha$ ; $Z_c = \frac{1}{j\omega C}$
- If there are sources with different frequencies, solve for each source, then summate time domain sinusoids (superposition)

- Impedances:

$$Z_L = j\omega L = jX_L \qquad Z_C = \frac{-j}{\omega C} = -jX_C \qquad Z_R = R$$

- Active Power: electrical power consumed by resistance (R)

$$P_{ave} = \frac{1}{T}\int_0^T v(t)\cdot i(t)dt = \frac{1}{R}\cdot\frac{1}{T}\int_0^T v^2(t)dt = R\cdot\frac{1}{T}\int_0^T i^2(t)dt$$

## Transfer Function: $h(t) \rightarrow H(s)$

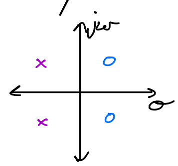

$\longrightarrow \frac{Y(s)}{X(s)} = H(s)$

- Assume Initial Conditions to compute
- Usually written as a ratio of polynomials of "s"
- Roots in numerator of transfer function result in zeros
  - HP-Prime "proot(numer(H(s))"
- Roots in denominator of transfer function result in poles
  - HP-Prime "proot(denom(H(s))"
- Stability

$$h(t) = u(t)e^{-\sigma_0 t} \xrightarrow{L} \frac{1}{s+\sigma_0} = H(s)$$

  - Stable if root is less than 0

## Complex Plane



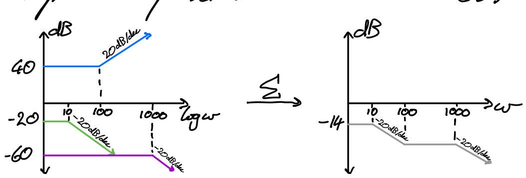

Zeros | Poles

As $\sigma$ becomes more negative, system decays faster: $e^{-at} = \frac{1}{s+a}$

- Decibel conversion ( $20\log_{10}|Y(s)|$ ) to apply transfer function:

$$20\log_{10}|Y(s)| = 20\log_{10}|H(s)| + 20\log_{10}|X(s)|$$

  - Graphed Representation: Ex: $H(s) = \frac{2(s+100)}{(s+10)(s+1000)}$




## Bode Plots

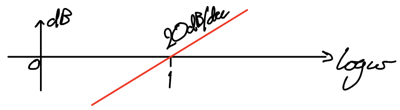
- decibels as a function of frequency: Exact dB $= 20\log_{10}|H(j\omega)|$

$$H(s) = \frac{a_n s^n + a_{n-1}s^{n-1} + \dots a_1 s + a_0}{s^m + b_{m-1}s^{m-1} + \dots b_1 s + b_0} = \frac{C s(s+z_1)(s+z_2)\dots(s+z_{n-1})}{(s+p_1)(s+p_2)\dots(s+p_m)}$$

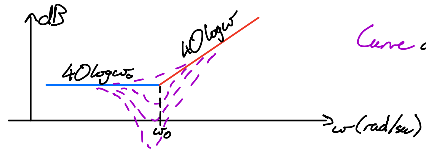
  - Represent as ratio of polynomials, or zeros and poles
  - Treat $(s-a)$ as $(s+a)$ → same contribution to dB Bode plot

$$H(s) = \frac{As\left(\frac{s}{z_1}+1\right)\left(\frac{s}{z_2}+1\right)\cdots\left(\frac{s}{z_{n-1}}+1\right)}{\left(\frac{s}{p_1}+1\right)\left(\frac{s}{p_2}+1\right)\cdots\left(\frac{s}{p_m}+1\right)} \;;\; \lim_{s\to\infty} H(s) = A$$

- Plot will be shifted up by: $\alpha_{dB} = 20\log_{10} A$
- A "bend" (zero or pole) will have maximum error of 3dB, with $\pm$ 20dB/dec
  - A higher order bend: $(s+p_1)^n$; $n>1$ will have maximum error of $n\cdot 3$dB, with $\pm n\cdot 20$dB/dec
- Zero at zero $\longrightarrow$ 0dB @ $\omega = 1$ rad/sec



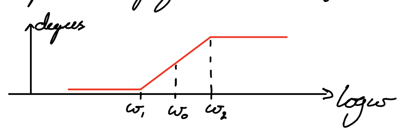

- Complex Conjugate: $(s^2+2\zeta\omega_c s+\omega_c^2) \longrightarrow$ contributes: $\alpha_{dB} = \sqrt{(\omega_c^2-\omega^2)^2+(2\zeta\omega_c\omega)^2}$
  - at high frequencies; $\omega \gg \omega_c$: $\alpha_{dB} \doteq 40\log_{10}\omega$
  - at low frequencies; $\omega \ll \omega_c$: $\alpha_{dB} \doteq 40\log_{10}\omega_0$




Curve depends on $\zeta$

- phase as a function of frequency: Exact $\Phi = \angle H(j\omega)$
  - Zero/Pole at $a$ contributes $\pm 45°$/dec at $0.1a$, and $\pm 45°$/dec at $10a$
  - +ve constant $\rightarrow$ 0° phase shift, −ve constant $\rightarrow$ −180° phase shift
  - Zero/Pole at 0 contributes $\pm 90°$, Treat $(s-a)$ as $s^2/(s+a)$ for phase Bode plot
- Complex Conjugate: $[s-(a+jb)][s-(a-jb)] = (s^2+2\zeta\omega_c s+\omega_c^2) \xrightarrow{s=j\omega} (\omega_c^2-\omega^2)+j2\zeta\omega_c\omega$

damping factor




$\omega_c = \sqrt{\omega_1\cdot\omega_2}$

$\text{slope}_{\omega_1\to\omega_2} = \frac{90°}{\zeta}$

$\omega_1 = \omega_c\cdot 10^{-\zeta}$

$\omega_2 = \omega_c\cdot 10^{\zeta}$

Exact Complex Conjugate $\Phi = \tan^{-1}\left(\frac{2\zeta\omega_c\omega}{\omega_c^2-\omega^2}\right)$

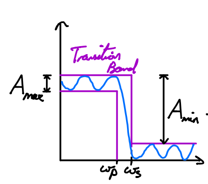
## Resonance

- Certain frequency that makes circuit look purely resistive from given port
- Bandwidth: width between half power frequencies
- Quality Factor: $Q = \omega_0/BW = \omega_2-\omega_1$ ; $Q \geq 10 \rightarrow$ highly selective filter

resonant frequency

## Filters

- Electric network that blocks unwanted frequencies
- Passband: signals that are allowed
- Stopband: signals that are not allowed



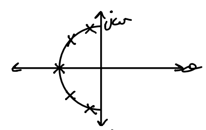

$\omega_p/\omega_s$ = Selectivity Factor

$A_{min} \rightarrow$ Passband Ripple

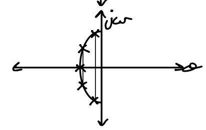
- Classification Types
  - Low Pass filter: $H(s) = 1/(s+a)$ ; below certain frequency
  - High Pass filter: $H(s) = s/(s+a)$ ; above certain frequency
  - Band Pass filter: $H(s) = s/(s+a)(s+b)$ ; range of certain frequencies
  - Band Stop filter: $H(s) = (s^2 + \omega_N^2)/(s^2 + BW \cdot s + \omega_N^2)$ ← Null Frequency

- Cutoff Frequency ($\omega_c$)
  - A pole that creates a bend from 0 slope to −ve slope or vice versa
  - Defined as $\omega_h$ & $\omega_L$ — frequencies in which the signal goes up/down 3dB from 0 slope
  - Occurs when $\omega_c$: $|H(j\omega_c)| = \frac{1}{\sqrt{2}} |H(j \cdot 0)|$
  - Canonical Form: $\omega_c = 1$ rad/s

- Butterworth Filters
  - All poles exist on circle around origin




- Chebyshev Filter
  - All poles exist in pairs, equidistant on ellipse around origin



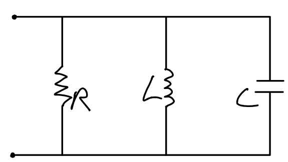

- Magnitude/Frequency Scaling
  - Increase resistors/inductors by factor $K_m$, and shift frequency response by $K_f$

$$R' = K_m \cdot R \qquad L' = \frac{K_m}{K_f} \cdot L$$

$$C' = \frac{C}{K_m \cdot K_f} \qquad \omega' = K_f \cdot \omega$$

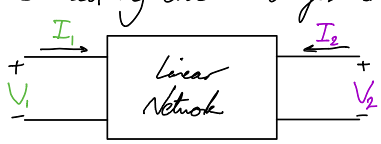
- RLC Circuit

$$\omega_0 = \frac{1}{\sqrt{LC}}, \quad Q = \omega_0 \cdot RC$$




## Two Port Network

- Characterized by two voltages and two currents → two as inputs/two as outputs




- Impedance Parameters $\left(Z = \frac{V}{I}\right)$

$$V_1 = Z_{11} I_1 + Z_{12} I_2 \qquad V_2 = Z_{21} I_1 + Z_{22} I_2$$

$$Z_{11} = \frac{V_1}{I_1} \quad Z_{21} = \frac{V_2}{I_1}$$

$$Z_{12} = \frac{V_1}{I_2} \quad Z_{22} = \frac{V_2}{I_2}$$

  - Network is Symmetric if $Z_{11} = Z_{22}$
  - Network is Reciprocal if $Z_{12} = Z_{21}$

- Admittance Parameters $\left(Y = \frac{I}{V}\right)$

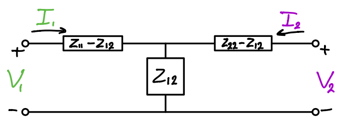
$$I_1 = Y_{11}V_1 + Y_{12}V_2 \qquad I_2 = Y_{21}V_1 + Y_{22}V_2$$

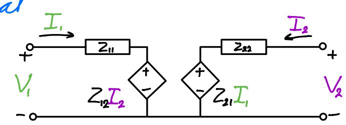
$$Y_{11} = \frac{I_1}{V_1} \qquad Y_{21} = \frac{I_1}{V_2}$$

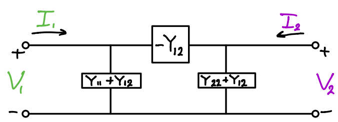
$$Y_{12} = \frac{I_2}{V_1} \qquad Y_{22} = \frac{I_2}{V_2}$$

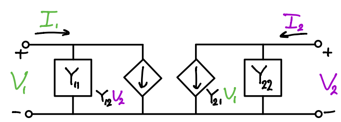
- Models of Network Cases

Reciprocal




- Hybrid Parameters (h)
  - Sometimes networks do not have exclusively admittance or impedance parameters

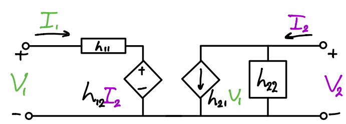
$$V_1 = h_{11}I_1 + h_{12}V_2 \qquad I_2 = h_{21}I_1 + h_{22}V_2$$

Short-circuit input impedance

Open-circuit reverse voltage gain

Short-circuit forward current gain

Open-circuit output admittance

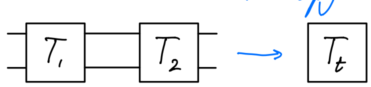



- Transmission Parameters $\left(T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)$

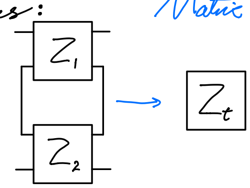
$$V_1 = AV_2 - BI_2$$
$$I_1 = CV_2 - DI_2 \longrightarrow \begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = [T]\begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

- Combine 2 Two Port Networks

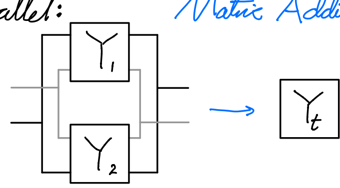
Cascade: Matrix Multiply




Series: Matrix Addition




Parallel: Matrix Addition



- Parameter Conversion Table

| to \ from | Z | Y | H | G | T |
|---|---|---|---|---|---|
| Z | $\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$ | $\begin{bmatrix} \frac{y_{22}}{\Delta_y} & \frac{-y_{12}}{\Delta_y} \\ \frac{-y_{21}}{\Delta_y} & \frac{y_{11}}{\Delta_y} \end{bmatrix}$ | $\begin{bmatrix} \frac{\Delta_h}{h_{22}} & \frac{h_{12}}{h_{22}} \\ \frac{-h_{21}}{h_{22}} & \frac{1}{h_{22}} \end{bmatrix}$ | $\begin{bmatrix} \frac{1}{g_{11}} & \frac{-g_{12}}{g_{11}} \\ \frac{g_{21}}{g_{11}} & \frac{\Delta_g}{g_{11}} \end{bmatrix}$ | $\begin{bmatrix} \frac{t_{11}}{t_{21}} & \frac{\Delta_t}{t_{21}} \\ \frac{1}{t_{21}} & \frac{t_{22}}{t_{21}} \end{bmatrix}$ |
| Y | $\begin{bmatrix} \frac{z_{22}}{\Delta_z} & \frac{-z_{12}}{\Delta_z} \\ \frac{-z_{21}}{\Delta_z} & \frac{z_{11}}{\Delta_z} \end{bmatrix}$ | $\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}$ | $\begin{bmatrix} \frac{1}{h_{11}} & \frac{-h_{12}}{h_{11}} \\ \frac{h_{21}}{h_{11}} & \frac{\Delta_h}{h_{11}} \end{bmatrix}$ | $\begin{bmatrix} \frac{\Delta_g}{g_{22}} & \frac{g_{12}}{g_{22}} \\ \frac{-g_{21}}{g_{22}} & \frac{1}{g_{22}} \end{bmatrix}$ | $\begin{bmatrix} \frac{t_{22}}{t_{12}} & \frac{-\Delta_t}{t_{12}} \\ \frac{-1}{t_{12}} & \frac{t_{11}}{t_{12}} \end{bmatrix}$ |
| H | $\begin{bmatrix} \frac{\Delta_z}{z_{22}} & \frac{z_{12}}{z_{22}} \\ \frac{-z_{21}}{z_{22}} & \frac{1}{z_{22}} \end{bmatrix}$ | $\begin{bmatrix} \frac{1}{y_{11}} & \frac{-y_{12}}{y_{11}} \\ \frac{y_{21}}{y_{11}} & \frac{\Delta_y}{y_{11}} \end{bmatrix}$ | $\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$ | $\begin{bmatrix} \frac{g_{22}}{\Delta_g} & \frac{-g_{12}}{\Delta_g} \\ \frac{-g_{21}}{\Delta_g} & \frac{g_{11}}{\Delta_g} \end{bmatrix}$ | $\begin{bmatrix} \frac{t_{12}}{t_{22}} & \frac{\Delta_t}{t_{22}} \\ \frac{-1}{t_{22}} & \frac{t_{21}}{t_{22}} \end{bmatrix}$ |
| G | $\begin{bmatrix} \frac{1}{z_{11}} & \frac{-z_{12}}{z_{11}} \\ \frac{z_{21}}{z_{11}} & \frac{\Delta_z}{z_{11}} \end{bmatrix}$ | $\begin{bmatrix} \frac{\Delta_y}{y_{22}} & \frac{y_{12}}{y_{22}} \\ \frac{-y_{21}}{y_{22}} & \frac{1}{y_{22}} \end{bmatrix}$ | $\begin{bmatrix} \frac{h_2 2}{\Delta_h} & \frac{-h_{12}}{\Delta_h} \\ \frac{-h_{21}}{\Delta_h} & \frac{h_{11}}{\Delta_h} \end{bmatrix}$ | $\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$ | $\begin{bmatrix} \frac{t_{21}}{t_{11}} & \frac{-\Delta_t}{t_{11}} \\ \frac{1}{t_{11}} & \frac{t_{12}}{t_{11}} \end{bmatrix}$ |
| T | $\begin{bmatrix} \frac{z_{11}}{z_{21}} & \frac{\Delta_z}{z_{21}} \\ \frac{1}{z_{21}} & \frac{z_{22}}{z_{21}} \end{bmatrix}$ | $\begin{bmatrix} \frac{-y_{22}}{y_{21}} & \frac{-1}{y_{21}} \\ \frac{-\Delta_y}{y_{21}} & \frac{-y_{11}}{y_{21}} \end{bmatrix}$ | $\begin{bmatrix} \frac{-\Delta_h}{h_{21}} & \frac{-h_{11}}{h_{21}} \\ \frac{-h_{22}}{h_{21}} & \frac{-1}{h_{21}} \end{bmatrix}$ | $\begin{bmatrix} \frac{1}{g_{21}} & \frac{g_{22}}{g_{21}} \\ \frac{g_{11}}{g_{21}} & \frac{\Delta_g}{g_{21}} \end{bmatrix}$ | $\begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}$ |

$$\Delta X = X_{11}X_{22} - X_{12}X_{21}$$

## AC Power Analysis

- Average Power

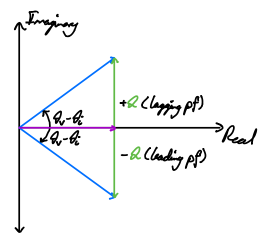
$$v(t) = V_m \cos(\omega t + \theta_v)$$
$$i(t) = I_m \cos(\omega t + \theta_i)$$

- Complex Power = Active Power + j · Reactive Power [VA]

$$S = P + jQ$$
$$= V_{rms} \cdot I_{rms}^{*} = I_{rms}^{2} \cdot Z$$

$$|S| = V_{rms} \cdot I_{rms} = \sqrt{P^2 + Q^2}$$
$$P = |S| \cos(\theta_v - \theta_i) \quad [W]$$
$$Q = |S| \sin(\theta_v - \theta_i) \quad [VAR]$$

Power Triangle




$$\text{Power Factor: } pf = \frac{P}{S} = \cos(\theta_v - \theta_i)$$

- Maximizing Power through Impedence (Z)
  - Thevenin equivalent impedence ($Z_{Th}$) around Z is complex conjugate of impedence value for Z as to draw maximum power:

$$Z = Z_{Th}^{*} = \left(\frac{V_{Th}}{i_{sc}}\right)^{*}$$

- Instantaneous Power [VA]

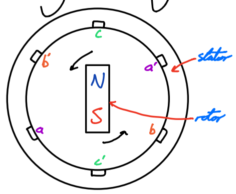
$$p(t) = V(t)i(t) = \frac{V_m i_m}{2}\left[\cos(\theta_v - \theta_i) + \cos(2\omega t + (\theta_v + \theta_i))\right]$$

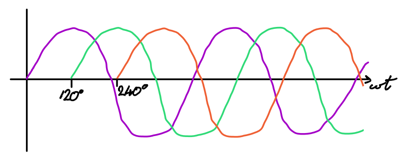
$$= \frac{V_m i_m}{2}\left[\underbrace{\cos(\theta_v - \theta_i)\cdot(1 + \cos(2\omega t))}_{P(t)} + \underbrace{\sin(\theta_v - \theta_i)\cdot \sin(2\omega t)}_{Q(t)}\right]$$

- Power Conservation
  - Sum of element powers of circuit must equal 0
  - All powers have power conservation except for Apparent Power [VA]

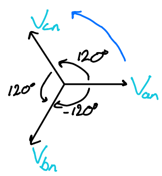
## Three Phase Circuits

- Circuit Network with three sources that are out of phase with each other by 120°, with same amplitude and frequency
- A three phase generator consists of a rotating magnet (rotor) surrounded by a stationary winding (stator)



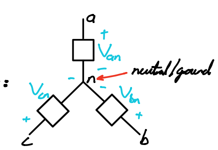

- Phase Sequence: time order in which the voltages pass through their maximum values




$V_{an} = V_p \angle 0°$
$V_{bn} = V_p \angle -120°$
$V_{cn} = V_p \angle 120°$

- Balanced Three-Phase Voltages are equal in magnitude and out of phase by 120°
- Balanced Load: impedances are equal in magnitude and phase
- Balanced Three-Phase Connections (Source - Load)

Y-configuration:

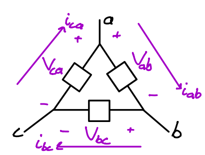



$i_a$

$3Z_Y = Z_\Delta$ ← Convert between Y & Δ

$i_b$

$i_c$




: Δ-Configuration

- Y-Y connection

Line Voltage

$V_L = V_p\sqrt{3}$

Phase Voltage

$V_{an} - V_{bn} = \sqrt{3}\, V_p \angle 30°$
$V_{bn} - V_{cn} = \sqrt{3}\, V_p \angle -90°$
$V_{cn} - V_{an} = \sqrt{3}\, V_p \angle -210°$

$V_L = |V_{ab}| + |V_{bc}| + |V_{ca}|$

$V_p = |V_{an}| + |V_{bn}| + |V_{cn}|$

Line Current

$i_L = i_p$

Phase Current

$i_a = V_{an}/Z_Y$
$i_b = V_{bn}/Z_Y$
$i_c = V_{cn}/Z_Y$

$i_a + i_b + i_c = 0$

- Y-Δ connection

$i_L = |i_a| + |i_b| + |i_c|$

$i_L = i_p\sqrt{3}$

$i_{ab} = V_{ab}/Z_\Delta$
$i_{bc} = V_{bc}/Z_\Delta$
$i_{ca} = V_{ca}/Z_\Delta$

$i_p = |i_{ab}| + |i_{bc}| + |i_{ca}|$

$V_L = V_p$

- Δ-Δ & Δ-Y connections
  - Convert these configurations to Y-Y

$V_{ab} = V_L$
$V_{bc} = V_L \angle -120°$
$V_{ca} = V_L \angle 120°$

$V_{an} = V_L/\sqrt{3} \angle -30°$
$V_{bn} = V_L/\sqrt{3} \angle -150°$
$V_{cn} = V_L/\sqrt{3} \angle 90°$

- Unbalanced Three-Phase Connections
  - Y-Y connection

$i_a = V_{an}/Z_a$ $\quad$ $i_a + i_b + i_c + i_n = 0$

$i_b = V_{bn}/Z_b$

$i_c = V_{cn}/Z_c$

- Instantaneous Power ($Z_Y = Z\angle\theta$)

$V_{an}(t) = V_p\sqrt{2}\cos(\omega t)$ $\quad$ $P_a(t) = 2V_p i_p \cos(\omega t)\cos(\omega t - \theta)$

$i_a(t) = i_p\sqrt{2}\cos(\omega t - \theta)$

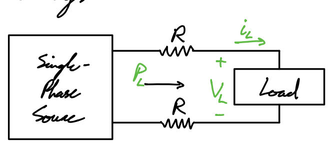
$$P(t) = P_a(t) + P_b(t) + P_c(t) = 3V_p i_p \cos\theta$$

  - Instantaneous Power as a function of time is a constant

- Power per Phase
  - Average: $P_p = V_p i_p \cos\theta$
  - Reactive: $Q_p = V_p i_p \sin\theta$
  - Apparent: $|S_p| = V_p i_p$
  - Complex: $S_p = V_p i_p \angle\theta = P_p + jQ_p = V_p \cdot i_p^*$ $\quad$ $S_L = 3i_L^2 \cdot Z_Y \angle\theta_{Z_Y}$

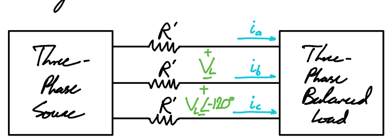
- Total Power
  - Average: $P = 3P_p = 3V_p i_p \cos\theta = \sqrt{3}V_L i_L \cos\theta$
  - Reactive: $Q = 3Q_p = 3V_p i_p \sin\theta = \sqrt{3}V_L i_L \sin\theta$
  - Apparent: $|S| = 3|S_p| = 3V_p i_p = \sqrt{3}V_L i_L$
  - Complex: $S = 3S_p = 3V_p i_p \angle\theta = \sqrt{3}V_L i_L \angle\theta = P + jQ = 3V_p \cdot i_p^*$

- Power Loss
  - Single-Phase System




$P_{Loss} = 2R\frac{P_L^2}{V_L^2}$

  - Three-Phase System

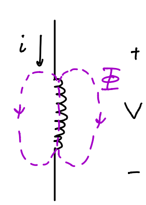



$P_{Loss} = R'\frac{P_L^2}{V_L^2}$

  - Three-Phase systems have much lower power loss

## Magnetically Coupled Circuits

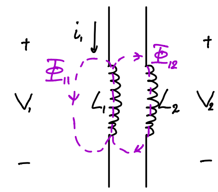
- Self-Inductance
  - Current passing through coil generates magnetic flux through it
  - A voltage is induced across the coil when flux changes




# of turns

$$V = N\frac{d\Phi}{dt} = N\frac{d\Phi}{di}\cdot\frac{di}{dt} = L\frac{di}{dt} \longrightarrow L = N\frac{d\Phi}{dt}$$

- Mutual-Inductance
  - Change in magnetic flux may result from nearby coil

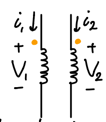



$$\Phi_1 = \Phi_{11} + \Phi_{12}$$

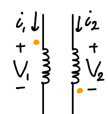
$$V_1 = N_1 \frac{d\Phi_1}{dt} \qquad V_2 = N_2 \frac{d\Phi_{12}}{dt_1} = M_{21}\frac{di_1}{dt}$$

$$M_{21} = N_2 \frac{d\Phi_{12}}{dt_1} \text{ [Henrys]}$$

  - Mutual-Inductance is the ability of one inductor to induce a voltage across a nearby inductor
  - Dot Convention
    - indicates polarity of mutual-inductance voltage




$\frac{V_2}{V_1} = \frac{N_2}{N_1}$ $\frac{i_2}{i_1} = -\frac{N_1}{N_2}$

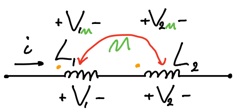



$\frac{V_2}{V_1} = -\frac{N_2}{N_1}$ $\frac{i_2}{i_1} = \frac{N_1}{N_2}$

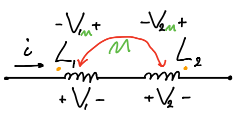
Dots on same side: $\hat{V}_1 = \hat{V}_2$ $\hat{i}_1 = -\hat{i}_2$

Dots on opposite side: $\hat{V}_1 = -\hat{V}_2$ $\hat{i}_1 = \hat{i}_2$

- Coupled Coils in Series




$$L_T = L_1 + L_2 + 2M \longrightarrow \text{Series Aiding Connection}$$




$$L_T = L_1 + L_2 - 2M \longrightarrow \text{Series Opposing Connection}$$

- Energy in Coupled Circuits

$$W = \frac{L_1 i_1^2}{2} + \frac{L_2 i_2^2}{2} \pm M i_1 i_2$$

+ if dots are on same side

− if dots are on opposite side

  - Energy Conservation bounds Mutual-Inductance

$$M \le \sqrt{L_1 L_2} \longrightarrow K = \frac{M}{\sqrt{L_1 L_2}} = \frac{\Phi_{12}}{\Phi_{11} + \Phi_{12}} \quad ; \quad 0 \le K \le 1$$

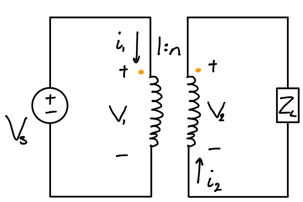
0: No magnetic flux crossover

1: Maximum magnetic flux crossover

    - K is a measure of how much magnetic flux passes from one coil to the other

- Ideal Transformer
  - Unity-coupled, lossless transformer with infinite self-inductances

$K = 1$ $R_1 = R_2 = 0\Omega$ $L_1 = L_2 = \infty$



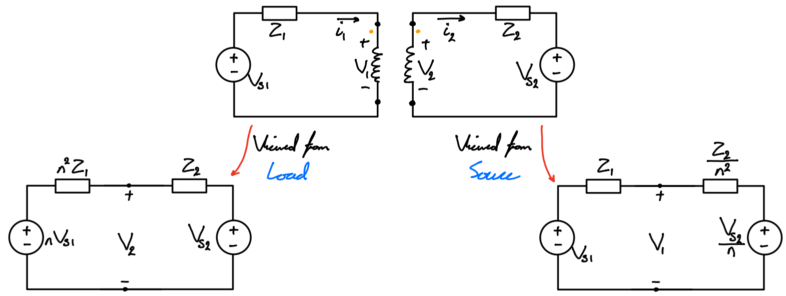

$n = \frac{N_2}{N_1}$ ← ratio of number of turns of inductors

$$\frac{V_2}{V_1} = \frac{N_2}{N_1} \qquad \frac{i_2}{i_1} = \frac{N_1}{N_2}$$

$V_2 > V_1 \longrightarrow$ step-up transformer

$V_2 < V_1 \longrightarrow$ step-down transformer

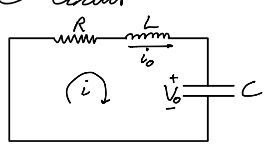
- Equivalent Circuit




## Second-Order Circuit

- A circuit characterized by a Second-Order Differential Equation
- Consists of 2 energy storage elements

- RLC Circuit




$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i\,dt = 0 \longrightarrow \text{KVL Equation}$$

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0 \longrightarrow \text{Second-Order Differential Equation}$$

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \longrightarrow \text{Characteristic Equation}$$

$$S_1 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

$$S_2 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$\alpha = \frac{R}{2L} \longrightarrow \text{Damping Factor}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \longrightarrow \text{Resonance Frequency}$$

- Solutions of Characteristic Equation (Transient Responses)

Determined by Initial Conditions

- Overdamped: $\alpha > \omega_0 \longrightarrow i(t)/v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$
- Critically Damped: $\alpha = \omega_0 \longrightarrow i(t)/v(t) = (A_1 t + A_2)e^{-\alpha t}$
- Underdamped: $\alpha < \omega_0 \longrightarrow i(t)/v(t) = A_1 e^{-(\alpha - j\omega_d)t} + A_2 e^{-(\alpha + j\omega_d)t}$

$$= e^{-\alpha t}[(A_1 + A_2)\cos(\omega_d t) + j(A_1 - A_2)\sin(\omega_d t)]$$

$$S_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2} = -\alpha \pm j\omega_d$$

damping frequency

- Steady-State Response: $V_{ss} = V(\infty) = V_s$ / $i_{ss} = i(\infty) = 0$
- Complete Solution = Steady-State Response + Transient Response

- Method to finding solution of second order circuits
  1. Transient Response by turning off independent sources
  2. Steady-State Response by finding final state