

- Signals: something that *carries/conveys* information
- Systems: something that *transforms/generates* signals
- Continuous-Time signal (analog) can be converted to Discrete-Time signal (digital) through *sampling* $\rightarrow$ a system

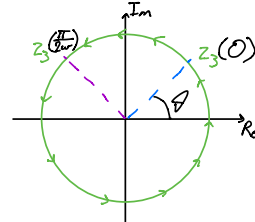
Complex Numbers

- Euler's Identity: $e^{j\theta} = \cos \theta + j \sin \theta$

- Rotating Complex Numbers

$$Z_1 = r e^{j\theta} \quad Z_2 = r e^{j\omega t}$$

$$Z_3 = Z_1 Z_2$$



- $\cos(\alpha + \beta)$ identity

$$= \operatorname{Re}(e^{j(\alpha + \beta)}) = \operatorname{Re}(e^{j\alpha} \cdot e^{j\beta}) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

Signals

- Signal Space is a set of all eligible signals:

$$X = [D \rightarrow R] = \{x \mid x: D \rightarrow R\}$$

single signal
mapping description

- Signal Classification

I. Deterministic	vs.	Stochastic
• Can be represented by formula		• Random signals
II. Continuous - Time		Discrete - Time
• Domain is set of $\mathbb{R}$		• Domain is set of $\mathbb{Z}$
III. Continuous - Amplitude		Discrete - Amplitude
• Range is set of $\mathbb{R}$		• Range is set of $\mathbb{Z}$
• $\mathbb{R}$ signals are <i>analog</i> , $\mathbb{Z}$ signals are <i>digital</i>		

- Basic Signal Operations

1. Amplitude Scaling

$$x(t) \rightarrow \triangle \rightarrow \alpha x(t)$$

2. Time Shift

• $\tau < 0 \rightarrow$ Right shift

$$x(t) \rightarrow \square \rightarrow x(t + \tau)$$

3. Time Scaling

- $|a| < 1 \rightarrow$ stretch
- $|a| = -1 \rightarrow$ reflection

$$x(t) \rightarrow \boxed{a} \rightarrow x(at)$$

4. Differentiation

$$x(t) \rightarrow \boxed{\frac{d}{dt}} \rightarrow \frac{dx}{dt}$$

5. Integration

$$x(t) \rightarrow \boxed{\int} \rightarrow \int_{-\infty}^t x(\tau) d\tau$$

6. Adder

$$x(t) \rightarrow \boxed{+} \rightarrow x(t) + y(t)$$

7. Time Windowing (Multiplying)

$$x(t) \rightarrow \boxed{\times} \rightarrow x(t)w(t)$$

IV. Even/Odd/Neither

- Even: $x(t) = x(-t)$
- Odd: $x(t) = -x(-t)$
- Decompose function into **even** and **odd** functions:

$$y_e(t) = 0.5(y(t) + y(-t))$$

$$y_o(t) = 0.5(y(t) - y(-t))$$

V. Periodic/Aperiodic

- defined for $t \in \mathbb{R}$ & there exists value T_0 such that:
 $x(t + kT_0) = x(t), k \in \mathbb{Z}$

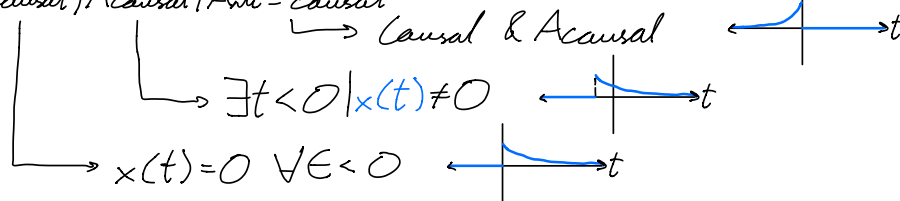
VI. Finite Energy (Decay)/Finite Power (Periodic)/Infinite Power

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$P_x = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt = \sum \frac{(\text{Amplitude})^2}{2}$$

- finite energy when signal energy (E_x) $< \infty \rightarrow P_x = 0$
- finite power when signal power (P_x) $< \infty$

VII. Causal/Acausal/Anti-causal



- Causal if output is function of only current and past inputs
- Acausal if output is function of past, present and/or future inputs

• Basic Signals

- Complex Exponential: $x(t) = Ae^{st}$; $s = \sigma + j\omega$

$$= |A|e^{\sigma t} [\cos(\Omega_0 t + \theta) + j \sin(\Omega_0 t + \theta)]; t \in \mathbb{R}$$

$$\Omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$$

ω_0 : fundamental angular frequency

• Unit Impulse: $\delta(t)$
 • has integral: $\int_{-\infty}^t \delta(t) dt = \begin{cases} 1, t > 0 \\ 0, t < 0 \end{cases} = u(t)$

• Unit Step: $u(t) = \begin{cases} 1, t > 0 \\ 0, t < 0 \end{cases} = \int_{-\infty}^t \delta(t) dt$

• Ramp Signal: $r(t) = t \cdot u(t)$, $r'(t) = u(t)$, $r''(t) = \delta(t)$

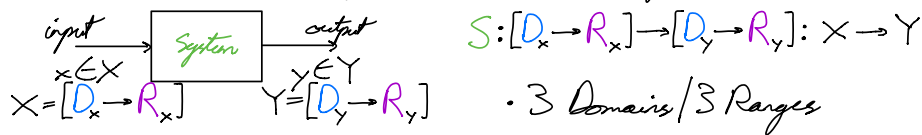
• Unit Pulse: $\text{rect}(t) = \begin{cases} 1, 0 \leq t \leq 1 \\ 0, \text{otherwise} \end{cases}$

• Impulse Train: $\delta_{T_s}(t) = \sum_{k=-\infty}^{\infty} \delta(t - T_s k)$; infinite impulses spaced " T_s " apart
Impulse Generator

• Sinc Function: $\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$; $t \in \mathbb{R}$

System

• a function that maps a domain signal into a range signal



• Classifications

• Memoryless: instantaneous output is only a function of input @ t

• Causal: instantaneous output is only a function of inputs up to t

• Linearity: system allows Homogeneity & Additivity

$$S(ax) = aS(x), \quad S(x+y) = S(x) + S(y)$$

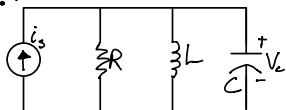
• Combined: $S(ax+by) = aS(x) + bS(y)$

• Time Invariant: output corresponding to shifted input is equal

$$\begin{aligned} x(t) &\rightarrow y(t) = S(x(t)) \\ x(t+\tau) &\rightarrow y(t+\tau) = S(x(t+\tau)) \end{aligned}$$

• Can apply to circuit elements:

Ex:



$$Z(s) = (R^{-1} + (sL)^{-1} + sC)^{-1} = \frac{V_L}{i_s}$$

$$\hookrightarrow (LCR \cdot s^2 + L \cdot s + R)V = (LR \cdot s)i_s$$

• ODE System Representation

- A linear ODE of any order is **Linear Time-Invariant**

$$a_0 + a_1 s + \dots + s^N = \prod_{k=1}^N (s - p_k) \quad \begin{array}{l} \text{polynomial roots /} \\ \text{natural frequencies /} \\ \text{eigenvalues} \end{array}$$

• Impulse Response: $h(t) = \mathcal{L}^{-1}\{H(s)\}$

- Output of system corresponding to $\delta(t)$, and initial conditions are 0
- LTI system has single unique impulse response

- Response of LTI system ($S(x) \Rightarrow h(t)$) to signal $x(t)$ is the convolution integral:

$$y(t) = S(x(t)) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t - \tau) d\tau = [x * h](t) = [h * x](t)$$

- Unit Step Response: $s(t) = \int_{-\infty}^{\infty} u(t - \tau) \cdot h(\tau) d\tau = \mathcal{L}^{-1}\{H(s) \cdot \frac{1}{s}\}$

- **Support** of Signal is region that is not zero

- Two LTI systems, with impulse response in **series** can be convolved and in **parallel**, can be summed

• ODE System Superposition

- Given linear ODE; a system's **Complete Response** ($y(t)$) has 2 components

- **Zero-State Response** / **Forced Response**

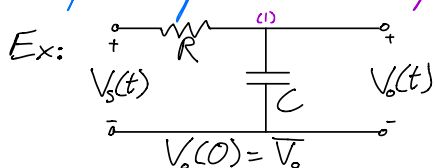
- $y_{zs}(t)$ is a function of input as initial conditions are 0

- **Zero-Input Response** / **Natural Response**

- $y_{zi}(t)$ is a function of initial conditions as input is 0

$$y(t) = y_{zs}(t) + y_{zi}(t)$$

- **Complete Response** = **Forced Response** + **Natural Response** = **Steady State** + **Transient**



$$K1: \frac{V_s - V_o}{R} = C V_o' \rightarrow V_o' + \frac{1}{RC} V_o = \frac{1}{RC} V_s \quad \leftarrow \text{ODE}$$

$$\hookrightarrow V_o(t) = A e^{-t/RC}$$

$$V_o(t) = V_s + (V_o - V_s) e^{-t/RC}$$

$$V_o(t) = V_s (1 - e^{-t/RC}), \quad V_o(t) = V_o e^{-t/RC}$$

• Second Order Linear ODE

$$\text{Given: } ay'' + by' + cy = f(t) \rightarrow \text{Solutions: } y(t) = y_h(t) + y_p(t)$$

$$\text{Homogeneous Solutions: } y_h'' + 2\omega_0 \zeta y_h' + \omega_0^2 y_h = 0$$

$$\hookrightarrow \text{Auxiliary Equation: } s^2 + 2\omega_0 \zeta s + \omega_0^2 = 0$$

- Roots will provide useful system information

- Overdamped: $\zeta > 1 \rightarrow y_h(t) = Ae^{s_1 t} + Be^{s_2 t}$
- Critically Damped: $\zeta = 1 \rightarrow y_h(t) = (A+Bt)e^{-\omega_s t}$
- Underdamped: $\zeta < 1 \rightarrow y_h(t) = e^{\sigma t} \cos(\omega_s t + \phi)$
 $\swarrow$
 $s = \sigma + j\omega_s$

- BIBO Stability
 - BIBO stable if impulse response is absolutely integrable: $\int_{-\infty}^{\infty} |h(t)| dt < \infty$
 - $j\omega$ -axis in RoC of $H(s)$
 - poles of $H(s)$ are in left s -plane (outputs will remain bounded for all time)

Laplace Representation

- One-Sided Laplace Transforms

$$\bar{f}(s) = \mathcal{L}\{f(t)u(t)\} = \int_0^{\infty} f(t)e^{-st} dt; \text{ RoC } \sigma > \sigma_1, \quad s = \sigma + j\omega$$

- Region of Convergence is the region encompassing all functions being transformed

- One-Sided Inverse Laplace Transform

$$f(t) = \mathcal{L}^{-1}\{\bar{f}(s)\} = \int_{\sigma_1 - j\omega}^{\sigma_1 + j\omega} \bar{f}(s) e^{st} ds$$

- For $F(s)$ of $f(t)$ to exist: $\left| \int_{-\infty}^{\infty} f(t) e^{-st} dt \right| \leq \left| \int_{-\infty}^{\infty} f(t) e^{-\sigma t} dt \right|$

- Partial Fraction Decomposition

- For highest degree pole: $K = \lim_{s \rightarrow r} (s-r) \bar{f}(s)$

- Extensions

- Determine end behaviour of Laplace function

- Initial Value Theorem (IVT)

- if: $\lim_{s \rightarrow \infty} (\gamma(s) \cdot s) = \text{constant}; \text{ True}$

- Final Value Theorem (FVT) / Steady State Response

- if: $\lim_{s \rightarrow 0} (\gamma(s) \cdot s) = \text{constant}; \text{ True}$

- Eigenfunction

- An input is called an Eigenfunction of an LTI system when:

$$y(t) = e^{s_0 t} \int_{-\infty}^{\infty} h(\tau) e^{-s_0 \tau} d\tau = H(s_0) x(t)$$

- Region of Convergence - Signal Support

Finite Support: Entire s -plane

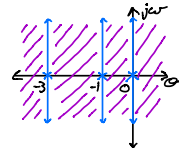
Causal Function: $\sigma > \max(\sigma_i), -\infty < \omega < \infty$

Anti-Causal: $\sigma < \min(\sigma_i), -\infty < \omega < \infty$

Non-Causal: $R = R_{\text{causal}} \cap R_{\text{anti-causal}}$

Ex: $G(s) = \frac{s^3 + 2s + 6}{s(s+1)^2(s+3)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s+1} + \frac{D}{(s+1)^2}$

4 Rd; each with their own solution



Fourier Series Representation

• Expansion of a periodic signal into a sum of sines and cosines

• Complex Exponentials Representation: $x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t}$

• Function set:

$\{e^{jk\omega_0 t}\}; k \in \mathbb{Z}$ provides orthonormal basis for all periodic functions

$$X_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) e^{-jk\omega_0 t} dt$$

$\omega_0 = \frac{2\pi}{T_0}$
 fundamental angular frequency
 fundamental period

• Trigonometric Representation:

$$x(t) = X_0 + 2 \sum_{k=1}^{\infty} |X_k| \cos(k\omega_0 t + \phi_k)$$

• $k=1 \rightarrow$ fundamental frequency;
 $k>1 \rightarrow$ harmonics

• Parseval's Power Relation: $P_x = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |X_k|^2$

• Coefficient Conjugates are equal: $X_k = X_{-k}^*$

• Arithmetic of Periodic Signals

• Addition:

$$Z_k = \alpha X_k + \beta Y_k$$

ratio of fundamental frequencies must be rational for sum to exist

$\frac{T_x}{T_y} = \frac{N}{M}$
 Fourier Series
 Coefficients of $x(t)$ & $y(t)$

• $z(t) = \alpha x(t) + \beta y(t)$ is periodic of fundamental period: $T_0 = M T_x = N T_y$

• Multiplication:

Greatest Common Factor of time frequencies

• $z(t) = x(t) \cdot y(t)$; frequency of $z(t)$ is: $\omega_z = \omega_x + \omega_y$

• Pure Sinusoid from Eigenfunctions

$$A \cos(\omega t + \theta) = \frac{A}{2} (e^{j(\omega t + \theta)} + e^{-j(\omega t + \theta)})$$

$$A \sin(\omega t + \theta) = \frac{A}{2j} (e^{j(\omega t + \theta)} - e^{-j(\omega t + \theta)})$$

• Use Eigenfunction representation to determine net fundamental frequency, and compute product of sinusoids

- Eigfunction Property
 • Response to a sinusoid / complex exponential can be found through:

$$x(t) = 2 \sum_{k=1}^{\infty} |X_k| \cos(k\omega_0 t + \angle X_k)$$

$$y(t) = 2 \sum_{k=1}^{\infty} |X_k| |H(jk\omega_0)| \cos(k\omega_0 t + \angle X_k + \angle H(jk\omega_0))$$

$$H(s) \Big|_{s=jk\omega_0}$$

Frequency Response of system at $\omega = k\omega_0$

- An input is called an Eigfunction of an LTI system when:

$$S[e^{j\omega t}] = H(j\omega) e^{j\omega t}$$

input signal can be multiplied by transfer function to get output

- Even / Odd Representation

- Even Periodic Signal: $x(t) = X_0 + 2 \sum_{k=1}^{\infty} X_k \cos(k\omega_0 t)$

- Fourier Coefficients of even signals (X_k) are Real

- Odd Periodic Signal: $x(t) = 2 \sum_{k=1}^{\infty} j X_k \sin(k\omega_0 t)$

- Fourier Coefficients of odd signals (X_k) are Imaginary

- Periodic Signal written as: $x(t) = x_e(t) + x_o(t)$ will have Fourier coefficients:

$$X_k = X_{ek} + X_{ok}$$

$$X_{ek} = \frac{1}{2} (X_k + X_{-k})$$

$$X_{ok} = \frac{1}{2} (X_k - X_{-k})$$

$x(-t)$ Fourier Coefficients

- Laplace Transform to Compute Fourier Coefficients:

$$X_k = \frac{1}{T_0} \cdot \mathcal{L}\{x(t)\}_{s=jk\omega_0}$$

Fourier Transform

- Decomposes a periodic signal into its constituent frequencies
- $X(\omega)$ describes how much of each frequency is in $x(t)$
- Bandwidth is: $\omega_{max} - \omega_{min} \rightarrow$ Sampling rate should be twice the Bandwidth
- Transform a time domain signal into a frequency domain signal

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \sum_k 2\pi X_k \delta(\omega - k\omega_0)$$

$$x(t) = \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$= \sum_k X_k e^{jk\omega_0 t}$$

- Fourier Transform and Inverse Fourier Transform are **Duals**: solving for one often allows for solving the other

- Sine Function: **sinc**(ω)

$$F\{u(t+1) - u(t-1)\} = e^{j\omega} - e^{-j\omega} = 2 \text{sinc}(\omega)$$

- Temporal Scaling: $F\{x(at)\} = \frac{1}{|a|} X(\frac{\omega}{a})$

- Frequency Scaling: $F^{-1}\{X(\beta\omega)\} = \frac{1}{|\beta|} x(\frac{t}{\beta})$

- Laplace Transform to Compute Fourier Transform:

$$X(\omega) = F\{x(t)\} = X(s)|_{s=j\omega}$$

- Parseval's Energy Relation:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

- Symmetry of Spectral Representations:

$$\begin{aligned} |X(\omega)| &= |X(-\omega)| \\ \angle X(\omega) &= -\angle X(-\omega) \\ \text{Re}[X(\omega)] &= \text{Re}[-X(-\omega)] \\ \text{Im}[X(\omega)] &= -\text{Im}[-X(-\omega)] \end{aligned}$$

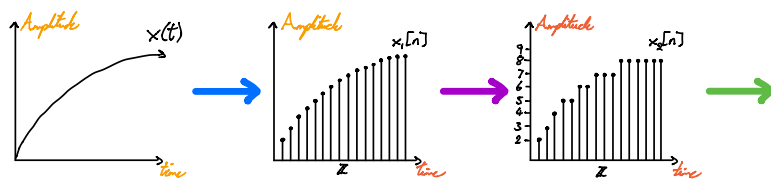
- Frequency Response: $H(\omega)$

$$Y(\omega) = H(\omega) X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi X_k H(jk\omega_0) \delta(\omega - k\omega_0)$$

- Fourier Transform of the Impulse Response gives the Frequency Response

Sampling Theory

- For a computer to process a signal, it must be **Discrete** in both Time and Amplitude
- **Sampling** is first step of Analog to Digital Converter
 - Converts **Continuous-Time** to **Discrete-Time**: capture Amplitude periodically
- **Quantization**
 - Converts **Continuous-Amplitude** to **Discrete-Amplitude**: map Amplitude to integer set
- **Encoding**
 - **Quantized** Amplitude is encoded into binary



Sampling a Sinusoid

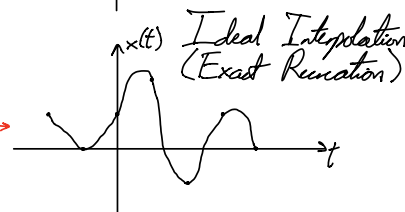
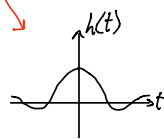
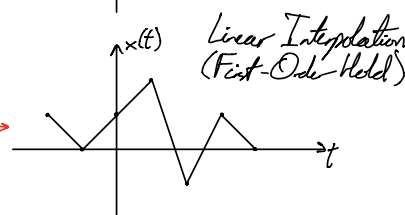
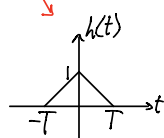
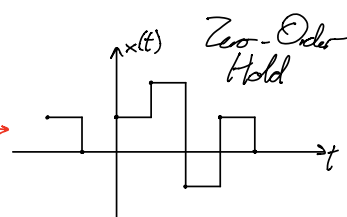
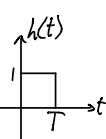
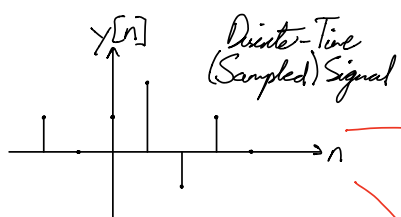
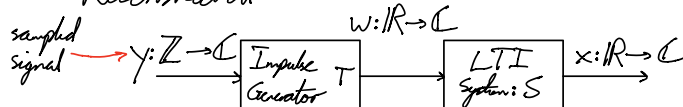
$$\forall t \in \mathbb{R}, x(t) = \cos(2\pi f_s t) \xrightarrow{\text{Sampler } T_s} \forall n \in \mathbb{Z}, y[n] = \cos(2\pi f_s n T_s)$$

$\forall$: for all instances

$$y[n] = x_c(t) \big|_{t=nT_s}$$

- Aliasing: an effect that causes different signals to become indistinguishable when sampling
- Occurs whenever the use of discrete elements to capture/produce a continuous-time signal causes **frequency ambiguity**

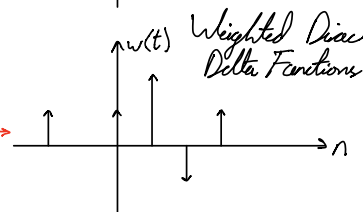
Reconstruction



$$h(t) = \frac{\sin(\pi t/T)}{\pi t/T} = \text{sinc}\left(\frac{t}{T}\right)$$

$$w(t) = \sum_{k=-\infty}^{\infty} y[k] \delta(t - kT)$$

result from an Ideal Impulse Generator



Nyquist-Shannon Sampling Theorem

- A band limited signal can be perfectly reconstructed from its samples if the average sampling rate satisfies the **Nyquist Sampling Rate Condition**:

$$\omega_s > 2\omega_{\max}$$

← highest frequency of input

Nyquist Rate is minimum sampling rate of given $\omega_{\max}$

- Given a sampling rate (ω_s), the **Nyquist Frequency** is the maximum input signal frequency that would still allow Ideal Interpolation: $\omega_s = 2\omega_{\max}$
- Pass sampled signal ($x_s[n]$) through ideal impulse generator cascaded to an ideal low pass filter with frequency response:

$$H(\omega) = \begin{cases} T_s, & -\omega_s/2 < \omega < \omega_s/2 \\ 0, & \text{otherwise} \end{cases}$$

$$= T_s \left[u(\omega + \frac{\omega_s}{2}) - u(\omega - \frac{\omega_s}{2}) \right]$$

- The reconstructed signal ($x_r(t)$):

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \frac{\sin[\frac{\pi}{T_s}(t-nT_s)]}{\frac{\pi}{T_s}(t-nT_s)}$$

Discrete-Time Signals/Systems

- Periodic Discrete-Time **sinusoids** are of the form: $x[n] = A \cos(\frac{2\pi m}{N}n + \theta)$; $-\infty < n < \infty$
- DT signal is periodic if:

- defined for $n \in (-\infty, \infty)$
- there is +ve integer N such that: $x[n+kN] = x[n]$

$$x(t) = A \cos(\omega_0 t + \theta) \xrightarrow[\omega_0 > 0, T_0 = \frac{2\pi}{\omega_0}]{\text{Sampler } T_s} x[n] = A \cos(\omega_0 T_s n + \theta) \quad \frac{T_s}{T_0} = \frac{m}{N}$$

← frequency multiplier

- To avoid information loss: $T_s < \frac{\pi}{\omega_0} = \frac{T_0}{2} \rightarrow$ Nyquist Sampling

- Sum: $z[n] = x[n] + y[n]$
 $N_x, N_y, N_z = qN_y + pN_x$; $\frac{N_y}{N_x} = \frac{p}{q}$ ← LCM is used for N_z

$$\text{Energy: } E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$\text{Power: } P_x = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N |x[n]|^2$$

- Even vs. Odd

$$\begin{aligned} \hookrightarrow x[n] &= -x[-n] \\ \hookrightarrow x[n] &= x[-n] \end{aligned}$$

$$x[n] = \frac{x[n] + x[-n]}{2} + \frac{x[n] - x[-n]}{2} = x_e[n] + x_o[n]$$

• Basic Signal Operations

• Unary Operators (one input)

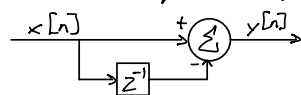
1. Amplitude Scaling $x(t) \rightarrow \alpha \rightarrow \alpha x(t)$

2. Time Shift $X[z], x(n) \rightarrow z^{-m} \rightarrow x(n-m), z^{-m}X[z]$
 • Delay by "m" steps where $m \in \mathbb{Z}$
 • Storing information for later release; stores "m" elements

3. Time Scaling $x[n] \rightarrow \alpha \rightarrow x[\frac{M}{L}n]; M, L \in \mathbb{Z}$
 • Downsampling: discarding data, lowering frequency (ω_s)
 • Upsampling: create data, increase frequency (ω_s)

4. Difference Operator

Backward Difference: $y[n] = x[n] - x[n-1]$

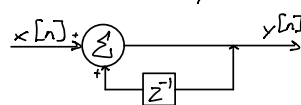


if $x[n] = v(nT_s)$, then $\left. \frac{dv(t)}{dt} \right|_{t=nT_s} \doteq \frac{y[n]}{T_s}$

• Forward Difference: $y[n] = x[n+1] - x[n]$

• Central Difference: $y[n] = \frac{x[n+1] - x[n-1]}{2}$

5. Accumulator: $y[n] = \sum_{k=-\infty}^n x[k] = y[n-1] + x[n]$



if $x[n] = v(nT_s)$, then $\int_{-\infty}^{nT_s} v(\tau) d\tau \doteq y[n] \cdot T_s$

• Binary Operators

1. Adder: $y[n] = x[n] + v[n]$

2. Time Multiplication: $y[n] = x[n] \cdot v[n]$

• C-Exponential

$A = |A|e^{j\theta}$, $\alpha = |\alpha|e^{j\Omega_0}$; $0 \leq \Omega_0 \leq \pi$

$x[n] = A\alpha^n = |A||\alpha|^n (\cos(\Omega_0 n + \theta) + j \sin(\Omega_0 n + \theta))$

• DT frequencies are effectively periodic in the frequency domain with period 2π

• Sinusoids

• Real Part of C-exponential: $x[n] = A \cos(\Omega_0 n + \theta) = A \frac{e^{j(\Omega_0 n + \theta)} + e^{-j(\Omega_0 n + \theta)}}{2}$
 $-\pi < \Omega \leq \pi$

• Unit Step & Impulse (Kronecker Delta)

$\delta[n] = \begin{cases} 1 & n=0 \\ 0 & \text{otherwise} \end{cases} = u[n] - u[n-1]$
 $u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} = \sum_{k=0}^{\infty} \delta[n-k]$
 (Backward Difference) $\rightarrow$ (Accumulator)



• Sifting Property: $x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n-k]$

• Any DT signal $x[n]$ is represented using unit-sample signals

• DT LTI Systems

• Linear if: $S(ax[n] + bv[n]) = aS(x[n]) + bS(v[n])$

• Time-Invariant if: $y[n \pm M] = S(x[n \pm M]); M \in \mathbb{Z}$

• Recursive (Autoregressive) & Non-Recursive

$y[n] = \sum_{m=0}^M b_m x[n-m] \rightarrow$ Finite Response: $y[n]$ is not dependent on $y[n-\infty]$

$\rightarrow y[n] = -\sum_{k=1}^M a_k x[n-k] + \sum_{m=0}^M b_m x[n-m] \rightarrow$ Infinite Response: $y[n]$ is dependent on $y[n-\infty]$

• Complete Response: $y[n] = y_{zi}[n] + y_{zs}[n]$

$y[n]|_{x[n]=0} = y_{zi}[n]$ $y[n]|_{y[k]=0 \forall k < 0} = y_{zs}[n]$

"n" if $h[n]$ is causal

• Convolution Sum: $y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] = [x * h][n]$

• BIBO Stability

• System is BIBO stable if system's impulse response is absolutely summable:

$$\sum_k |h[n]| < \infty$$

Ex: Input frequency: 96Hz $x(t) = A \cos(2\pi(96)t + \phi)$
Sampling frequency: 20Hz

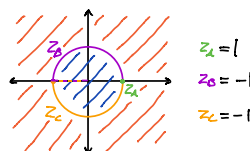
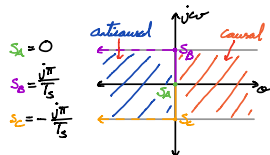
$x(t)$ sampled at 20Hz looks the same as $v(t) = A \cos(2\pi f t + \phi)$
 f can equal 76, 56, 36, 16, -4; must be: $f < \frac{20}{2}$

— Z-Transforms

• Convert DT-domain equations into complex frequency domain equations

$$Z\{x[n]\} = L\{x(t)\}_{z=e^{sT_s}} = \sum_n x[n] \cdot z^{-n} = X(z)$$

$$z = e^{sT_s} = e^{(\sigma + j\omega)T_s} = r e^{j\omega T_s}$$



• $-\frac{\pi}{T_s} \leq \omega \leq \frac{\pi}{T_s}$ in the s -plane maps to the entire z -plane
• Frequencies outside this range are aliased to lower planes

• Region of Convergence

• R_oC is the area in the z -domain for which the summation is finite

$$|X(z)| = \left| \sum_n x[n] z^{-n} \right| < \infty$$

$$R_oC(x) = \{z = r e^{j\omega} \in \mathbb{C} \mid \sum_n x[n] r^{-n} < \infty\}$$

- Poles of a z -transform $X(z)$ are $\mathbb{C}$ -values $\{p_n\}$ such that: $X(p_n) \rightarrow \infty$
- Zeros of a z -transform $X(z)$ are $\mathbb{C}$ -values $\{z_n\}$ such that: $X(z_n) = 0$
- Two-Sided z -transforms' R_oC is contained within poles

• Acausal convolution

• if $x[n]$ is acausal for $n < -N_1 < 0$, and $h[n]$ is acausal for $n < -N_2 < 0$ then:

$$y[n] = [x * h][n] = y[n + N_2 + N_1]$$

• Inverse Z -Transform

• When $X(z) = B(z)/A(z)$ is expressed as: $X(z) = \sum_{k=0}^{\infty} x[k] z^{-k}$ then dividing $B(z)$ by $A(z)$ gives: $\sum_{k=0}^{\infty} x[k]$ and $x[n] = \sum_{k=0}^{\infty} x[k] \delta[n-k]$

$$x[n] = \frac{1}{2\pi j} \oint_{\text{contour within } R_oC} X(z) z^{n-1} dz$$

• Transfer Function of FIR Systems (Non-Recursive)

$$y[n] = \sum_{k=0}^m b_k x[n-k] \rightarrow h[n] = \sum_{k=0}^m b_k \delta[n-k]$$

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^m b_k z^{-k} = \frac{1}{z^m} \sum_{k=0}^m b_k z^{m-k}$$

all poles at origin $\rightarrow$ BIBO stable

• Transfer Function of IIR Systems (Recursive)

$$y[n] = \sum_{k=0}^m b_k x[n-k] - \sum_{m=1}^M b_m y[n-m]$$

$$h[n] = Z^{-1} \left\{ \frac{B(z)}{A(z)} \right\} = \sum_{l=0}^{\infty} h[l] \delta[n-l]$$

• BIBO stable if poles are within unit circle

• Steady-State / Transient Responses

- Poles *inside* unit circle give Transient Response (in Left-Half-Plane)
- Poles *outside* or *on* unit circle give Increasing Response
- Poles *on* the $j\omega$ -axis give Steady-State Response

DT Fourier

• CT vs DT

Periodic: Fourier Series $\leftrightarrow$ Discrete Fourier Series / Discrete Fourier Transform

Aperiodic: Fourier Transform $\leftrightarrow$ Discrete-Time Fourier Transform

DFS/DFT

- Finite or Periodic function represented by an infinite number of sinusoids
- Since function is Discrete, its Fourier Series Coefficients are periodic and represented by a finite set of complex numbers $\rightarrow$ DFT

$$X_{\text{DFT}}[k] = \sum_{n=0}^{N-1} x[n] e^{jk\Omega_s n}$$

$\Omega_s = \frac{2\pi}{N}$
 n time steps of one period

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_{\text{DFT}}[k] e^{jk\Omega_s n}$$

$$X_{\text{DFS}}[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{jk\Omega_s n}$$

$$x[n] = \sum_{k=0}^{N-1} X_{\text{DFS}}[k] e^{jk\Omega_s n}$$

DTFT

- Produces a function of frequency that is a periodic summation of the CT Fourier Transform

$$X_{\text{DTFT}}(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

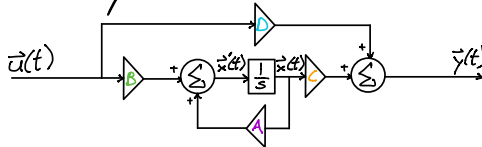
$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_{\text{DTFT}}(\Omega) e^{j\Omega n} d\Omega$$

Frequency Response

$$H(\Omega) = H^*(\Omega) = Z\{h[n]\}_{z=e^{j\Omega}}$$

State Space Representation

- replace n^{th} order differential equation with a single first order matrix differential equation
- State: set of first order differential equations written in terms of state variables
 - can determine $[t_0, \infty)$ of system provided state at t_0
- CT System State-Space LTI Model



$\vec{u}(t)$: input vector with "p" inputs

$\vec{x}(t)$: "N" dimensional state vector for N^{th} order system

$\vec{y}(t)$: output vector with "q" outputs

$$\dot{\vec{x}}(t) = \mathbf{A}\vec{x}(t) + \mathbf{B}\vec{u}(t) \rightarrow \text{State Differential Equation}$$

$\mathbf{A}$: describes stability of system $\rightarrow$ System Matrix: $\mathbf{A} \in \mathbb{R}^{N \times N}$

$\mathbf{B}$: describes how state can be controlled $\rightarrow$ Input Matrix: $\mathbf{B} \in \mathbb{R}^{N \times p}$

$$\vec{y}(t) = \mathbf{C}\vec{x}(t) + \mathbf{D}\vec{u}(t) \rightarrow \text{Response Equation}$$

$\mathbf{C}$: describes outputs of state $\rightarrow$ Output Matrix: $\mathbf{C} \in \mathbb{R}^{q \times N}$

$\mathbf{D}$: describes direct links between input and output $\rightarrow$ Feedforward Matrix: $\mathbf{D} \in \mathbb{R}^{q \times p}$

- State Transition Matrix ($e^{\mathbf{A}t}$)

used to compute various responses:

$\vec{y}(t)$ & $\dot{\vec{x}}(t)$ comprise the System State-Space Representation

• State Response: $\vec{x}(t) = e^{At} \vec{x}_0 + \int_0^t e^{A(t-\tau)} B \vec{u}(\tau) d\tau$ Identity Matrix

$$\mathcal{L}\{\vec{x}(t)\} = (sI - A)^{-1} \vec{x}_0 + (sI - A)^{-1} B \vec{U}(s)$$

• Output Response: $\vec{y}(t) = C e^{At} \vec{x}_0 + C \int_0^t e^{A(t-\tau)} B \vec{u}(\tau) d\tau + D \vec{u}(t)$

$$\mathcal{L}\{\vec{y}(t)\} = C (sI - A)^{-1} \vec{x}_0 + \underbrace{[C (sI - A)^{-1} B + D]}_{\text{Transfer Function: } \vec{H}(s)} \vec{U}(s)$$

• Controllability

- ability to drive systems from one state to another in a finite time interval

$$\vec{x}(t_f) = e^{A t_f} \vec{x}_0 + \int_0^{t_f} e^{A(t_f-\tau)} B \vec{u}(\tau) d\tau$$

- a system is completely controllable if $\vec{u}$ exists satisfying an arbitrary state $\vec{x}(t_f)$

• (A, B) is controllable when: rank of $\underbrace{[B, AB, \dots, A^{N-1}B]}_{\text{Controllability Matrix: } M_c} = N$

• Observability

- ability to infer initial state from observing input and output in a finite time interval

$$\vec{y}(t) = C e^{At} \vec{x}_0 + C \int_0^t e^{A(t-\tau)} B \vec{u}(\tau) d\tau + D \vec{u}(t)$$

- a system is completely observable if $\vec{x}_0$ can be determined given $\vec{y}(t)$ and $\vec{u}(t)$

• (A, C) is observable when: rank of $\underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{N-1} \end{bmatrix}}_{\text{Observability Matrix: } M_o} = N$

• Similarity Transformation

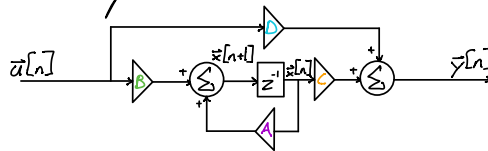
- the state-space representation is not unique

$$\text{Let } \vec{x}(t) = T \vec{\tilde{x}}(t) \rightarrow \begin{aligned} \dot{\vec{\tilde{x}}}(t) &= \tilde{A} \vec{\tilde{x}}(t) + \tilde{B} \vec{u}(t) \\ \vec{y}(t) &= \tilde{C} \vec{\tilde{x}}(t) + \tilde{D} \vec{u}(t) \end{aligned}$$

$$[A, B, C, D] \rightarrow [\tilde{A} = TAT^{-1}, \tilde{B} = TB, \tilde{C} = CT^{-1}, \tilde{D} = D]$$

- $\vec{\tilde{x}}(t)$ is another valid state vector for system based on transformation matrix: $T \in \mathbb{R}^{N \times N}$

• DT System State-Space LTI Model



• State Transition Matrix (A^n)

- used to compute various responses:

• State Response: $\vec{x}[n] = A^n \vec{x}_0 + \sum_{k=0}^{n-1} A^{n-1-k} B \vec{u}[k]$

$$Z\{\vec{x}[n]\} = (zI - A)^{-1} \vec{x}_0 + (zI - A)^{-1} B \vec{U}(z)$$

• Output Response: $\vec{y}[n] = C A^n \vec{x}_0 + \sum_{k=0}^{n-1} C A^{n-1-k} B \vec{u}[k] + D \vec{u}[n]$

$$Z\{\vec{y}[n]\} = C (zI - A)^{-1} \vec{x}_0 + \underbrace{[C (zI - A)^{-1} B + D]}_{\text{Transfer Function: } \vec{H}(z)} \vec{U}(z)$$

$\vec{x}[n+1] = A \vec{x}[n] + B \vec{u}[n] \rightarrow \text{State Difference Equation}$

$\vec{y}[n] = C \vec{x}[n] + D \vec{u}[n] \rightarrow \text{Response Equation}$

• Controllability

• ability to drive systems from one state to another in N steps (or less)

• (A, B) is controllable when: rank of $[M_c] = N$

• Observability

• ability to infer initial state from observing N inputs or outputs (or less)

• (A, C) is observable when: rank of $[M_o] = N$

• Computation of State Transition Matrix

• Eigendecomposition of A :

$$A = [\vec{v}_1 \dots \vec{v}_N] \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_N \end{bmatrix} \begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_N \end{bmatrix}^{-1} = T D T^{-1}$$

Eigenvalues Eigenvectors

CT: $e^{At} = T \begin{bmatrix} e^{\lambda_1 t} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{\lambda_N t} \end{bmatrix} T^{-1} = T e^{Dt} T^{-1}$

DT: $A^n = T \begin{bmatrix} \lambda_1^n & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_N^n \end{bmatrix} T^{-1} = T D^n T^{-1}$

• Cayley-Hamilton Theorem:

• Any function of A can be written as $f(A) = \sum_{k=0}^{N-1} c_k A^k$

CT: $e^{At} = \sum_{k=0}^{N-1} c_k A^k$, $e^{\lambda_i t} = \sum_{k=0}^{N-1} c_k \lambda_i^k$

DT: $A^n = \sum_{k=0}^{N-1} c_k A^k$, $\lambda_i^n = \sum_{k=0}^{N-1} c_k \lambda_i^k$