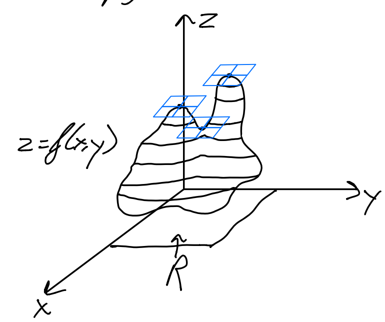
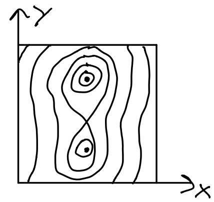
Multivariable Extrema

- Observe Critical Points
- Look along Boundary Points
- Apply 2nd derivative test to Critical Points



Contour Plots



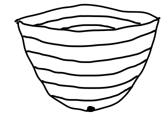
- (x_0,y_0) is a critical point if $\vec{\nabla} f(x_0,y_0)=\vec{0}$
  $f_x(x_0,y_0)=f_y(x_0,y_0)=0$;
  tangent plane at (x_0,y_0) is horizontal; $z=z_0$

- Critical Point Behaviours

local max

$z=-x^2-y^2$

local min



$z=x^2+y^2$

saddle point

$z=x^2-y^2$

- Determine which behaviour by examining

$$D=\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}\cdot f_{yy}-f_{xy}^2$$

1) $D(x_0,y_0)<0 \rightarrow (x_0,y_0)$ is saddle point
2) $D(x_0,y_0)>0 \rightarrow (x_0,y_0)$ is local max/min
   i) $f_{xx}(x_0,y_0)>0 \rightarrow$ local min
   ii) $f_{xx}(x_0,y_0)<0 \rightarrow$ local max

Ex: Let $f(x,y)=(2x-x^2)(2y-y^2)$, find and classify all critical points

$\vec{\nabla} f=\vec{0}$

$f_x=(2-2x)(2y-y^2)$ $\quad f_x=0$ when $x=1$ & $y=0,2$

$f_y=(2x-x^2)(2-2y)$ $\quad f_y=0$ when $x=0,2$ & $y=1$

$\therefore$ critical points at: $(1,1),(0,0),(0,2),(2,0),(2,2)$

$f_{xx}=-2(2y-y^2)$

$f_{yy}=-2(2x-x^2)$

$f_{xy}=(2-2x)(2-2y)$

$D(x_0,y_0)=f_{xx}(x_0,y_0)\cdot f_{yy}(x_0,y_0)-f_{xy}(x_0,y_0)^2$

$D(1,1)=4$; min/max $\rightarrow f_{xx}(1,1)=-2$; max

$D(0,0)=-4$; saddle

$D(0,2)=4$; min/max $\rightarrow f_{xx}(0,2)=0$; undefined

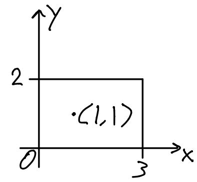
$D(2,0)=4$; min/max $\rightarrow f_{xx}(2,0)=0$; undefined

$D(2,2)=-4$; saddle

Find max/min of $f(x,y) = (2x - x^2)(2y - y^2)$ on domain:

$f(3,y) = -3(2y - y^2) = -6y + 3y^2 = g(y)$ $\quad S = \{0 \le x \le 3,\ 0 \le y \le 2\}$

find extrema of $g(y)$, $0 \le y \le 2$



$g'(y) = 6y - 6 \rightarrow y = 1$

$g''(y) = 6 > 0 \rightarrow y = 1$ is local min of $g(y)$

$g(1) = -3$ is minimum

$\therefore (1,1)$ is global max

$(3,1)$ is global min

Ex: $f(x,y) = -x^2 - y^3 + 3y^2$ on $\mathbb{R}^2$, what is the absolute max/min?

there is no absolute min due to cubic function

Critical Points:

$f_x = -2x \longrightarrow x = 0$

$f_y = -3y^2 + 6y \longrightarrow y = 0, 2$

Classification:

$f_{xx} = -2$

$f_{yy} = -6y + 6 = -6(y-1)$

$f_{xy} = 0$

$D = 12(y-1)^2$

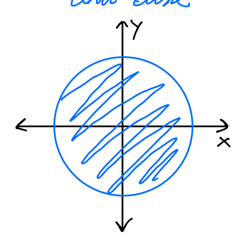
$D(0,0) = -12 < 0 \longrightarrow$ saddle point $(0,0)$

$D(0,2) = 12 > 0$

$f_{xx} = -2 < 0 \longrightarrow$ local max $(0,2)$

• Find the extrema of $f(x,y) = -x^2 - y^3 + 3y^2$ on Domain of $R = \{(x,y) \in \mathbb{R}^2,\ \underbrace{x^2 + y^2 \le 1}\}$

unit disk



$(0,0)$ is saddle point $\rightarrow$ not extrema

$(0,2)$ is outside boundary $\rightarrow$ not in domain

$\therefore$ extrema must occur on: $x^2 + y^2 = 1$

Domain $f(x,y) \longrightarrow g(y) = -y^3 + 4y^2 - 1$

extrema of $g(y)$ on $1 \le y \le 1$

CP: $0 = g'(y)$; $y = 0$, $y = 8/3 \rightarrow$ outside of Domain

$g''(0) = 8 > 0$; local min

$g(0) = -1 \longrightarrow$ minimum of $g(y)$

$\therefore$ boundaries must be maximum

$f(\pm 1, 0) = g(0) = -1 \longrightarrow$ minimum $(\pm 1, 0)$

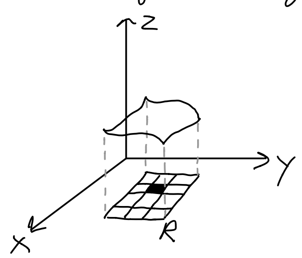
$f(0,-1) = g(-1) = 4 \longrightarrow$ maximum $(0,1)$

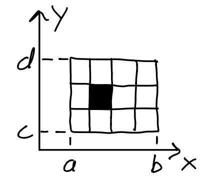
— Lagrange Multiplier − λ

$$\vec{\nabla} f(x_0, y_0) = \lambda \vec{\nabla} g(x_0, y_0)$$ where $f(x,y)$ is constrained by $g(x,y)$

$$\left.\begin{aligned} f_x &= \lambda g_x \\ f_y &= \lambda g_y \\ g &= 0 \end{aligned}\right\} \text{solve for } x, y, \lambda$$

— Surface Integration





Approximate volume under graph:
(divided into $N \times M$ intervals)

$$\sum_{i=1}^{N} \sum_{j=1}^{M} f(x_i, y_j) \Delta x \Delta y$$

$$\iint_R f(x,y)\,dx\,dy = \lim_{\substack{N \to \infty \\ M \to \infty}} \sum_{i=1}^{N} \sum_{j=1}^{M} f(x_i, y_j) \Delta x \Delta y$$

- Average of surface over $R = \dfrac{1}{N \cdot M} \displaystyle\sum_{i=1}^{N} \sum_{j=1}^{M} f(x_{ij}^*, y_{ij}^*) \Delta x \Delta y$

$$= \frac{1}{(b-a)(d-c)} \iint_R f(x,y)\,dx\,dy$$

- Partial Integration

Using Iterated Integration: $\displaystyle\iint_R f(x,y)\,dx\,dy = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$

- Fubini's Theorem

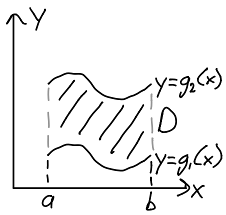
$$\iint_R f(x,y)\,dx\,dy = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

Ex: $\displaystyle\int_0^1\int_0^1 x(y+x^2)^2\,dx\,dy = \int_0^1 \left(\frac{1}{6}(y+x^2)^3 \Big|_0^1 \right) dy = \frac{1}{6}\left[\frac{1}{4}(y+1)^4 - \frac{1}{4}y^4 \right]_0^1$

$= 7/12$

- Integrate over any Domain

Type 1: $D = \{(x,y): a \le x \le b, g_1(x) \le y \le g_2(x)\}$

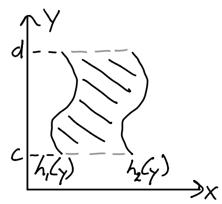


$$\iint_D f(x,y)\,dA = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right) dx$$

Ex: $\iint_D (1-x^2)^{3/2}\,dA$, $D = \{(x,y): x^2+y^2 \le 1\}$

$$= \int_{-1}^{1} \left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1-x^2)^{3/2}\,dy \right) dx = 2\int_{-1}^{1} (1-x^2)^{3/2}\,dx = \frac{32}{15}$$

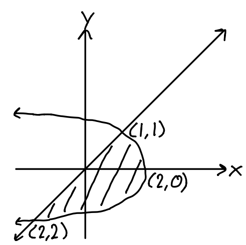
Type 2: $D = \{(x,y): c \le y \le d, h_1(y) \le x \le h_2(y)\}$



$$\iint_D f(x,y)\,dA = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx \right) dy$$

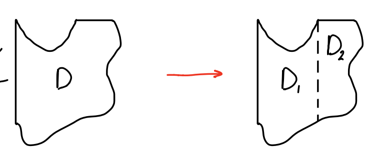
Ex: Let D be closed bounded region between $x=y$ & $x = 2-y^2$, find: $\iint_D y\,dA$

$$\int_{-2}^{1} \left(\int_{y}^{2-y^2} y\,dx \right) dy = -\frac{9}{4}$$



- Separating Domain

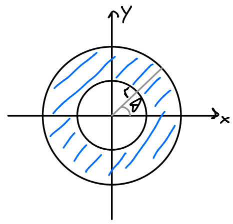
This domain is neither Type 1 or Type 2



$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$

- Represent Integral Limits with Polar Coordinates

Ex: $D = \{(x,y): 1 \leq x^2+y^2 \leq 4\} \rightarrow \{(r,\theta): 1 \leq r \leq 2,\ 0 \leq \theta \leq 2\pi\}$



$x \rightarrow r\cos\theta$

$y \rightarrow r\sin\theta$

$dxdy \rightarrow r\,dr\,d\theta$

$$\iint_D f(x,y)\,dxdy \rightarrow \int_1^2\int_0^{2\pi} r\cdot f(r\cos\theta, r\sin\theta)\,dr\,d\theta$$

Applications of Integrals in $\mathbb{R}^3$

- Mass/Electric Charge/etc. of volume described by D with density $\rho(x,y)$:

$$M/Q/etc. = \iint_D \rho(x,y)\,dxdy$$

- Centre of Mass/Electric Charge/etc.; $(\bar{x}, \bar{y})$:

$$\bar{x} = \frac{\iint_D x\cdot\rho(x,y)\,dxdy}{\iint_D \rho(x,y)\,dxdy}$$

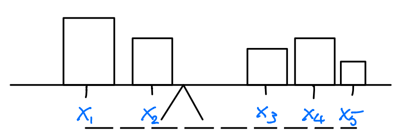
Moment about y-axis

$$\bar{y} = \frac{\iint_D y\cdot\rho(x,y)\,dxdy}{\iint_D \rho(x,y)\,dxdy}$$

Moment about x-axis

- Derived using sum of Torques: $m_1 d_1 = m_2 d_2$

$$\rightarrow \sum_i (x_i - \bar{x})m_i = 0 \longrightarrow \int_a^b (x-\bar{x})\rho(x)\,dx = 0$$



- Moment of Inertia

$$I_x = \iint_D y^2 f(x,y)\,dx\,dy \qquad I_y = \iint_D x^2 f(x,y)\,dx\,dy$$

$$I_0 = I_x + I_y$$

- Surface Area

of $g(x,y,z) = \iint_D \sqrt{\frac{g_x^2 + g_y^2 + g_z^2}{g_z^2}}\,dx\,dy$

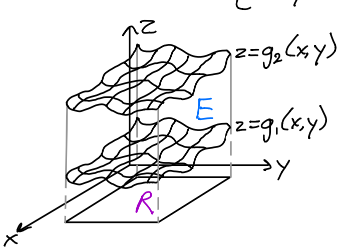
of $f(x,y) = \iint_D \sqrt{f_x^2 + f_y^2 + 1}\,dx\,dy$

—— Triple Integrals

- function $f(x,y,z)$ bounded by 3 space solid E

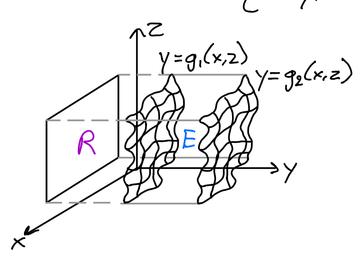
$$\iiint_E f(x,y,z)\underbrace{dx\,dy\,dz}_{dV}$$

- Type I $\quad E = \{(x,y,z) : (x,y) \in R,\ g_1(x,y) \leq z \leq g_2(x,y)\}$



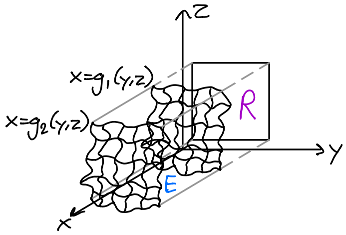
$$\iint_R \left[\int_{g_1}^{g_2} f(x,y,z)\,dz\right] dA$$

- Type II $\quad E = \{(x,y,z) : (x,z) \in R,\ g_1(x,z) \leq y \leq g_2(x,z)\}$



$$\iint_R \left[\int_{g_1}^{g_2} f(x,y,z)\,dy\right] dA$$

- Type III $E = \{(x,y,z) : (y,z) \in R, g_1(y,z) \le x \le g_2(y,z)\}$



$$\iint_R \left[\int_{g_1}^{g_2} f(x,y,z)\,dx \right] dA$$

- There are 6 possible ways to describe E, (Type I & II for each triple integral type)
- Represent Triple Integral with Cylindrical Coordinates

$$\iiint_E f(x,y,z)\,dx\,dy\,dz = \iiint_E f(r\cos\theta, r\sin\theta, z)\,dz\,dr\,d\theta$$

- Represent Triple Integral with Spherical Coordinates

$$\iiint_E f(x,y,z)\,dx\,dy\,dz = \iiint_E f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$