Laplace Transforms

- Definition: $L\{f(t)\} = \bar{f}(s) = \int_0^\infty e^{-st} \cdot f(t)\,dt$

$$L\left\{\frac{t^{n-1}}{(n-1)!}\right\} = \frac{1}{s^n}$$

$$L\left\{\frac{e^{-at}t^{n-1}}{(n-1)!}\right\} = \frac{1}{(s+a)^n}$$

$$L\{\cos\omega t\} = \frac{s}{s^2+\omega^2}$$

$$L\{\sin\omega t\} = \frac{\omega}{s^2+\omega^2}$$

$$L\{e^{at}f(t)\} = \bar{f}(s-a)$$

$$L\{H(t-a)f(t-a)\} = e^{-sa}\bar{f}(s)$$

$$L\{\delta(t-a)f(t)\} = f(a)e^{-sa}$$

$$L\{t^k f(t)\} = (-1)^k \frac{d}{ds^k}\bar{f}(s)$$

$$L\left\{\frac{dy}{dt}\right\} = s\bar{y}(s) - y(0)$$

$$L\left\{\frac{d^2y}{dt^2}\right\} = s^2\bar{y}(s) - sy(0) - y'(0)$$

$$L\{f+g\} = \bar{f} + \bar{g}$$

$$L\{H(t-a)f(t)\} = e^{-as}L\{f(t-a)\}$$

Ex: $ay'' + by' + cy = f(t)$

$$L\{ay'' + by' + cy\} = \bar{f}(s)$$

$$a\left(s^2\bar{y}(s) - sy(0) - y'(0)\right) + b\left(s\bar{y}(s) - y(0)\right) + c\bar{y} = \bar{f}(s)$$

$$(as^2 + bs + c)\bar{y} = \bar{f}(s) + ay(0) + ay'(0) + by(0)$$

$$\bar{y} = \frac{\bar{f}(s) + ay(0) + ay'(0) + by(0)}{as^2 + bs + c}$$

Ex: $y'' - 2y' - 3y = 0$, $y(0) = 0$, $y'(0) = 1$

$$s^2\bar{y} - 1 - 2s\bar{y} - 3\bar{y} = 0$$

$$\bar{y} = \frac{0 + 0 + 1 + 0}{s^2 - 2s - 3} = \frac{A}{s-3} + \frac{B}{s+1}$$

$$1 = A(s+1) + B(s-3)$$

$$\begin{aligned} 1 &= A - 3B \\ 0 &= A + B \end{aligned} \longrightarrow A = \frac{1}{4},\ B = -\frac{1}{4}$$

$$\bar{y} = \frac{1}{4(s-3)} - \frac{1}{4(s+1)}$$

$$y(t) = \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t}$$

- Shifting Theorem

1) $s \to s-a$: $\mathcal{L}\{e^{at}f(t)\} = \bar{f}(s-a)$

2) Step Function: $\mathcal{L}\{H(t-a)f(t-a)\} = e^{-sa}\bar{f}(s)$

Ex: $\frac{e^{-sa}}{s} = e^{-sa}\cdot\frac{1}{s} = e^{-sa}\cdot\bar{f}(s)$

$\therefore \mathcal{L}^{-1}\{\frac{e^{-sa}}{s}\} = H(t-a)$

Ex: $\frac{\omega}{\omega^2+s^2}\cdot e^{-sa} = \bar{f}(s)\cdot e^{-sa}$

$\therefore \mathcal{L}^{-1}\{\frac{\omega}{\omega^2+s^2}\cdot e^{-sa}\} = \sin[\omega(t-a)]\cdot H(t-a)$

Ex: $y''+y = H(t)-H(t-1)$, $y(0)=y'(0)=0$

$\mathcal{L}\{y''+y = H(t)-H(t-1)\} \longrightarrow (s^2+1)\bar{y} = \frac{1}{s} - \frac{e^{-s}}{s}$

$\bar{y} = (1-e^{-s})\cdot\frac{1}{s(s^2+1)} = (1-e^{-s})\cdot\left(\frac{1}{s} - \frac{s}{s^2+1}\right)$

$\mathcal{L}^{-1}\{\frac{1}{s} - \frac{s}{s^2+1}\} = 1-\cos t$, $\mathcal{L}^{-1}\{\frac{e^{-s}}{s} - \frac{se^{-s}}{s^2+1}\} = H(t-1)[1-\cos(t-1)]$

$\therefore y(t) = 1-\cos t - H(t-1)[1-\cos(t-1)]$

- Delta Functions

$$\int_a^b f(t)\delta(t-t_0)\,dt = \begin{cases} f(t_0), & a<t_0<b \\ 0, & t_0\leq a \,\|\, t_0\geq b \end{cases}$$

$$\int_{-\infty}^{t} \underbrace{\delta(\tau-t_0)}_{\text{impulse}}\,d\tau = \underbrace{H(t-t_0)}_{\text{switch}} \qquad \mathcal{L}\{\delta(t-t_0)f(t)\} = f(t_0)e^{-st_0}$$

Ex: An ODE with an impulse: $y''+\omega^2 y = \delta(t-a)$, $a>0$, $y(0)=y'(0)=0$

$\mathcal{L}\{\delta(t-a)\} = e^{-sa}$

$\mathcal{L}\{y''+\omega^2 y = \delta(t-a)\} = (s^2+\omega^2)\bar{y} = e^{-sa} \longrightarrow \bar{y} = \frac{e^{-sa}}{s^2+\omega^2}$

$\therefore y(t) = \frac{H(t-a)}{\omega}\sin[\omega(t-a)]$

- Convolutions

convolution of $f(t)$ & $g(t) = \int_0^t f(t-\tau)\cdot g(\tau)\,d\tau = f*g$

$\mathcal{L}\{f*g\} = \bar{f}(s)\cdot\bar{g}(s)$

$$y(t) = \mathcal{T} * g + C \leftarrow \text{dependent on initial conditions}$$

$$L\{\mathcal{T}\} = \frac{1}{as^2+bs+c}$$

- If $F(s) = L\{f(t)\}$ then $s \cdot F(s) = L\{f'(t) + f(0)\}$

Fourier Series

- Periodic Functions

  $y(x)$ with period of T

- Periodic Boundary Conditions for 2^{nd} Order ODE

  $ay'' + by' + cy = f(t)$

  Solved on $x=\alpha$ to $x=\beta$, where $T = b-a$, Boundary Conditions: $y(\alpha) = y(\beta)$, $y'(\alpha) = y'(\beta)$

  Solution is periodic with period T

- if $f(x)$ is periodic, then: $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx))$ ← integer

  where a_0, a_n, b_n are constants

$$\text{for } \int_0^{2\pi} f(x)dx = \int_0^{2\pi} \frac{a_0}{2} + 0\,dx \longrightarrow a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx \quad , \quad L = 2\pi$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx$$

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{mx\pi}{L}\right)dx$$

$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{mx\pi}{L}\right)dx$$

$$f_{even}(x) \longrightarrow b_n = 0$$

$$f_{odd}(x) \longrightarrow a_0 = a_n = 0$$

- Useful Integrals

$$\int_0^{2\pi} \cos(nx)\sin(mx)dx = 0$$

$$\int_0^{2\pi} \cos(nx)\cos(mx)dx = \begin{cases} 0, & n \neq m \\ \frac{2\pi}{2}, & n = m \end{cases}$$

$$\int_0^{2\pi} \sin(nx)\sin(mx)dx = \begin{cases} 0, & n \neq m \\ \frac{2\pi}{2}, & n = m \end{cases}$$

- Solving ODE: $y'' - y = f(x)$, $y(-\pi) = y(\pi)$, $y'(-\pi) = y'(\pi)$

  Ex: $y'' - y = \sum_{n=1}^{\infty} a_n \cos(nx)$ $\qquad a_n = \frac{2}{\pi n^2}\left[1-(-1)^n\right]$

  Particular

  Try: $y_p = \sum_{n=1}^{\infty} d_n \cos(nx) \longrightarrow y_p'' = \sum_{n=1}^{\infty} (-n^2 \cdot d_n) \cos(nx)$

$$\sum_{n=1}^{\infty} (-n^2 \cdot d_n - d_n)\cos(nx) = \sum_{n=1}^{\infty} a_n \cos(nx)$$

$$(n^2+1)d_n = -a_n$$

$$d_n = \frac{-a_n}{n^2+1} = -\frac{2}{(n^2+1)\pi n^2}\left[1-(-1)^n\right]$$

  Homogeneous

$$m^2 - 1 = 0 \longrightarrow e^{\pm x}$$

$$y = A_1 e^{x} + A_2 e^{-x} - \sum_{n=1}^{\infty} \frac{2}{(n^2+1)\pi n^2}\left[1-(-1)^n\right]\cos(nx)$$

- Odd and Even Functions
  - Square Wave: odd function $\longrightarrow \int_{-\pi}^{\pi} f(x)dx = 0$
  - Sawtooth Wave: even function $\longrightarrow \int_{-\pi}^{\pi} f(x)dx = 2\int_0^{\pi} f(x)dx$
  - if $f(x)$ is even:

$$\int_{-\pi}^{\pi} f(x)\cdot\cos(nx)dx = 2\int_0^{\pi} f(x)\cdot\cos(nx)dx = a_n$$

$$\int_{-\pi}^{\pi} f(x)\cdot\sin(nx)dx = 0 = b_n$$

  - if $f(x)$ is odd:

$$\int_{-\pi}^{\pi} f(x)\cdot\cos(nx)dx = 0 = a_n$$

$$\int_{-\pi}^{\pi} f(x)\cdot\sin(nx)dx = 2\int_0^{\pi} f(x)\cdot\cos(nx)dx = b_n$$

Partial Differential Equations

- A function that depends upon: $u', \frac{\delta u}{\delta x}, \frac{\delta u}{\delta y}, \frac{\delta^2 u}{\delta x^2}, \frac{\delta^2 u}{\delta y^2}, \dots, x, y$
  where $u(x,y)$ is the solution

- 3 Common Types:

Diffusion $\longrightarrow \frac{\delta u}{\delta t} = K \frac{\delta^2 u}{\delta x^2}$, $u(x,t)$

Laplace $\longrightarrow \frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0$, $u(x,y)$

Wave $\longrightarrow \frac{\delta^2 u}{\delta t^2} = c^2 \frac{\delta^2 u}{\delta x^2}$, $u(x,t) = F(x - c \cdot t) + G(x + c \cdot t)$

arbitrary functions

- General Solution for: $\frac{\delta u}{\delta t} = \frac{\delta^2 u}{\delta x^2}$ (Using Separation of Variables)

$$u(x,t) = X(x) \cdot T(t) \longrightarrow \begin{matrix} u_t = XT' \\ u_{xx} = X''T \end{matrix} \longrightarrow \frac{X''}{X} = \frac{T'}{T} = -\lambda$$

separation constant

$$\longrightarrow T = Ce^{-\lambda t}, \; X = \begin{cases} A\cos\sqrt{\lambda}\, t \\ B\sin\sqrt{\lambda}\, t \end{cases} \xrightarrow[\text{Conditions}]{\text{Boundary}} \lambda = \left(\frac{n\pi}{L}\right)^2$$

Given: $u_t = u_{xx}$ with $u(x,0) = f(x)$
$0 < x < \pi$ $\quad u(0,t) = u(\pi,t) = 0 \longrightarrow X(0) = X(\pi) = 0$
$t > 0$

$X(0) = 0 \longrightarrow A = 0$
$X(\pi) = 0 \longrightarrow B\sin(\sqrt{\lambda}\,\pi) = 0$, true for all $\lambda = n^2$, where $n = 1, 2, 3 \dots$

$\longrightarrow u(x,t) = \underset{b_n}{\cancel{BC}}\, e^{-n^2 t} \sin nx$, where a_0, a_n, b_n, n are defined by $f(x)$

the general solution will be the sum of all solutions differentiated by "n"

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin nx \cdot e^{-n^2 t} \longrightarrow f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

extended to $-\pi < x < \pi$ as odd function $\longrightarrow b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$

- General Solution for: $\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0$, $u_x(0,y) = u_x(L,y) = 0$, $u(x,0) = f(x)$

$X'' + \lambda X = 0 \longrightarrow X = \begin{cases} A\cos\sqrt{\lambda}\, x \\ B\cos\sqrt{\lambda}\, x \end{cases}$ $\qquad A\sqrt{\lambda}\sin(\sqrt{\lambda} L) = 0$, $B = 0$

$Y'' - \lambda Y = 0 \longrightarrow Y = \begin{cases} Ce^{\sqrt{\lambda} y} \\ De^{-\sqrt{\lambda} y} \end{cases}$ $\qquad \hookrightarrow X = A\cos\left(\frac{n\pi x}{L}\right)$ &

$$Y = Ce^{-\frac{n\pi y}{L}} + De^{\frac{n\pi y}{L}}, \; \lambda = \frac{n^2 \pi^2}{L^2}$$

$$u = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n e^{-\frac{n\pi y}{L}} + \cancel{\hat{a}_n e^{\frac{n\pi y}{L}}}\right]\cos\left(\frac{n\pi x}{L}\right)$$

Due to initial condition: $(u_y \to 0, y \to \infty)$, this is impossible to solve unless $\hat{a}_n = 0$

even function

$$u(x,0) = f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

- Inhomogeneous Boundary Conditions
  - Separation of Variables method does not work here, so use the steady state solution to make Boundary Conditions homogeneous
    $U(x) = \alpha x + \beta$, $\beta = U(0)$, $L\alpha = U(L) - \beta$

Given: $u_t = u_{xx}$, $u(0,t) = c_1$, $u(L,t) = c_2$, $u(x,0) = f(x)$

Set: $u(x,t) = U(x) \longrightarrow \beta = c_1$, $\alpha = \frac{c_2 - \beta}{L} \longrightarrow U(x) = \frac{c_2 - c_1}{L}x + c_1$

Set: $u(x,t) = U(x) + v(x,t)$

$\therefore v_t = v_{xx}$, $v(0,t) = v(L,t) = 0$, $v(x,0) = f(x) - U(x)$

Then solve for $v(x,t)$ using Separation of Variables

- Inhomogeneous PDE
  - Use steady state solution

Given: $u_t = u_{xx} + 1$, $u(0,t) = u(1,t) = 0$, $u(x,0) = f(x)$

Steady State: $U_t = U_{xx} + 1 \longrightarrow U'' = -1$, $U(1) = U(0) = 0$

$\longrightarrow U(x) = \frac{x}{2}(1-x)$

$\therefore v_t = v_{xx}$, $v(0,t) = v(1,t) = 0$, $v(x,0) = f(x) - \left(\frac{x}{2}(1-x)\right)$

Then solve for $v(x,t)$ using Separation of Variables