Parameterization

- Surfaces: $S = r(x,y) = \langle x, y, f(x,y) \rangle$ where $(x,y) \in D$

Ex: $x^2+y^2=z$, $z \le 9 \longrightarrow r(b,\theta) = \langle \sqrt{b}\cos\theta, \sqrt{b}\sin\theta, b \rangle$,
where: $0 \le b \le 9$, $0 \le \theta \le 2\pi$

$r(x,y) = \langle x, y, x^2+y^2 \rangle$,
where: $(x,y) \in D$, $D: \{x^2+y^2 \le 3\}$

- Curves
  - Circle: $x^2+y^2=a^2$; $r(t) = \langle a\cos t, a\sin t \rangle$; $0 \le t \le 2\pi$

  Ex: $(x-1)^2+y^2=1$; $r(t) = \langle \cos t + 1, \sin t \rangle$; $0 \le t \le 2\pi$

  - Ellipse: Seperate y radius and x radius

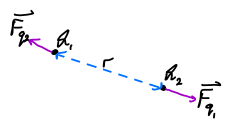
  $r(t) = \langle a\cos t, b\sin t \rangle$; $0 \le t \le 2\pi$

  - Cylinder Helix: $r(t) = \langle a\cos t, a\sin t, \frac{bt}{2\pi} \rangle$; where b is period length of helix

Coulomb's Law

$$F_q = \frac{q_1 q_2}{4\pi\varepsilon_0 \cdot r^2}$$

Electric Permittivity in Vacuum: $\frac{1}{36\pi}E\text{-}9 \frac{C}{m^2}$

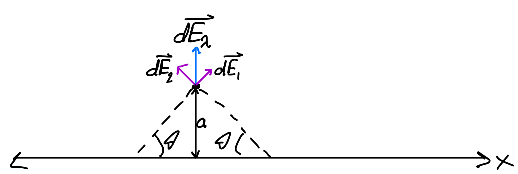


- Permittivity: physical quantity describing how an electric field responds to a dielectric medium
- Electric Field at any point created by q:

$$E_q = \frac{q}{4\pi\varepsilon_0 \cdot r^2}$$

  - Line of Charge ($\lambda = q/x$):

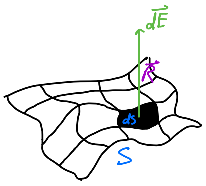
$$E_\lambda = \frac{\lambda}{2\pi\varepsilon_0 a}$$



  - Plane of Charge ($\sigma = q/xy$)

$$E_\sigma = \frac{\sigma}{2\varepsilon_0}$$

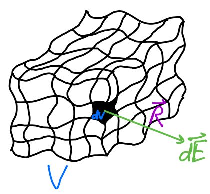
- Surface of Charge ($\sigma = Q/xy$)



$$d\vec{E} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{dQ}{|\vec{R}|^2} \cdot \hat{R} \quad , \; dQ = \sigma \, dS$$

$$\vec{E} = \frac{\sigma}{4\pi\varepsilon_0} \iint_S \frac{dS}{|\vec{R}|^3} \cdot \vec{R}$$

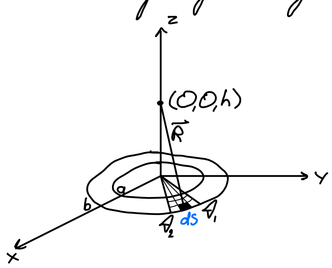
- Volume of Charge ($\rho = Q/xyz$)



$$d\vec{E} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{dQ}{|\vec{R}|^2} \cdot \hat{R} \quad , \; dQ = \rho \, dV$$

$$\vec{E} = \frac{\rho}{4\pi\varepsilon_0} \iiint_V \frac{dV}{|\vec{R}|^3} \cdot \vec{R}$$

- Rings of Charge



$$dS = r \, dr \, d\theta$$

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{dQ}{r^2+h^2} \cdot \hat{R} \quad , \; dQ = \sigma \, dS = \sigma r \, dr \, d\theta$$

$$\vec{E} = \frac{\sigma}{4\pi\varepsilon_0} \iint_S \frac{r}{(r^2+h^2)^{3/2}} \cdot \vec{R} \, dr \, d\theta = \frac{\sigma \cdot h}{4\pi\varepsilon_0} \int_0^{2\pi} \int_a^b \frac{r}{(r^2+h^2)^{3/2}} \cdot dr \, d\theta \cdot \hat{k}$$

Work

$$dW = -q \, \vec{E} \cdot d\vec{r} \longrightarrow W = -q \int_{\text{initial}}^{\text{final}} \vec{E} \cdot d\vec{r} \quad \text{Joules}$$

final/initial is the distance between charges

- Adding work to system: $W > 0$
- Removing work from system: $W < 0$

$$dW_c = |\vec{F}_c| \cdot ds$$

force along path "C" ⟶ "L(t)" is parametrized "C"

$$= \vec{F}(L(t)) \cdot \frac{dL}{dt} \longrightarrow W_c = \int_{t_i}^{t_f} \vec{F}(L(t)) \cdot \frac{dL}{dt} \, dt$$

Potential

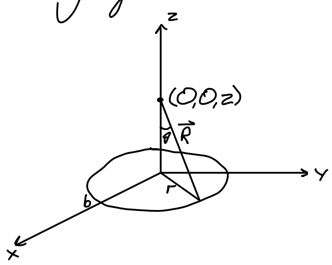
$$V_{\text{final}} - V_{\text{initial}} = -\int_{\text{initial}}^{\text{final}} \vec{E} \cdot d\vec{r} \quad \text{Volts}$$

- ΔV is positive if work moves charges closer

- Static Electic Field
  - A conservative field; path independent

$$\oint \vec{E}\cdot d\vec{r} = 0 \longrightarrow \text{Kirchoff's Voltage Law for closed loop}$$

- Ring of Potential



$$\vec{E}(z) = \frac{Qz}{4\pi\varepsilon_0 (r^2+z^2)^{3/2}} \cdot \hat{k} \quad \frac{N}{m}$$

$$V = -\int_{initial}^{final} \vec{E}\cdot d\vec{r} = -\int_{\infty}^{z} \frac{Qz}{4\pi\varepsilon_0 (r^2+z^2)^{3/2}} \cdot \hat{k} \cdot dr\,\hat{k}$$

$$= \frac{Q}{4\pi\varepsilon_0 \sqrt{r^2+z^2}} \quad \text{Volts}$$

- difference in voltage between point of interest, and infinity

Line Integral

- The line integral of a vector field $\vec{F}(x,y,z)$ on a curve $C: \vec{r}(t)$ is defined by:

$$\longrightarrow \int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot \vec{r}'(t)\,dt =$$

Gradient Fields

- Electric field created by a charge distribution is a gradient field
- There exists a scaler function V such that:

$$\vec{E} = -\vec{\nabla} V$$
$$\langle E_x, E_y, E_z \rangle = -\langle \frac{\delta V}{\delta x}, \frac{\delta V}{\delta y}, \frac{\delta V}{\delta z} \rangle$$

  - To solve for V, determine all terms that could be excluded by partial:

$$V = \int E_x\,dx + C(y,z)$$

- Work done by a gradient field does not depend on path
- If $\vec{G} = \vec{\nabla} g \longrightarrow$ the second partial derivatives are equal, then the Curl equals 0
- If $\vec{\nabla}\times\vec{G} \neq 0$; then $\vec{G}$ is not a gradient field, otherwise $\vec{G}$ could be a gradient field
- Curl is related to rotation of gradient field

Gauss' Law

- Electric Flux Density ($\vec{D}$)

$$\Phi = \oiint_S \vec{D}\cdot d\vec{s} = Q_{enclosed} = \vec{G}\cdot\hat{n}\cdot S \qquad \vec{D} = \frac{\Phi}{S} = \varepsilon\cdot\vec{E},\ \varepsilon = \varepsilon_r\cdot\varepsilon_0$$

Field · Normal of Surface · Area of Surface · Relative Permittivity

- Total flux through surface is equivalent to charge enclosed by surface
- A ball of charge can be treated as point charge
- Flux through any surface:

$$\Phi = \pm\iint_D \vec{G}\cdot\underbrace{\left(\frac{\delta\vec{r}}{\delta u}\times\frac{\delta\vec{r}}{\delta v}\right)dS}_{\hat{n}\,d\vec{r}(u,v)} = \iiint_V \nabla\cdot\vec{D}\,dV$$

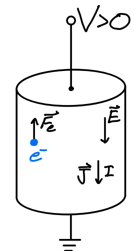
parametrized domain · $du\,dv$ · equals volume charge density: ρ_v

- Divergence can be used to determine flux through surface of volume V

Conductors

$$\vec{F_e} = -e\cdot\vec{E} \qquad \vec{V_d} = -\mu_e\cdot\vec{E}$$

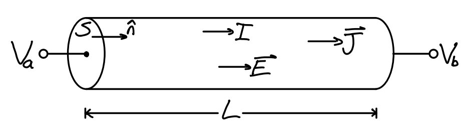
drift velocity · mobility constant



current density

$$\vec{J} = \rho_v\cdot\vec{v} = \rho_e\cdot\vec{V_d} = -\rho_e\cdot\mu_e\cdot\vec{E} = \sigma\cdot\vec{E}\ \left[\frac{\text{Ampere}}{\text{meter}^2}\right]$$

$$I = \iint_S \vec{J}\cdot d\vec{s} = \iint_S \vec{J}\cdot\hat{n}\,ds \qquad V_{ab} = -\int_b^a \vec{E}\cdot\hat{n}\,dL = E\cdot L \longrightarrow E = \frac{V_{ab}}{L}$$



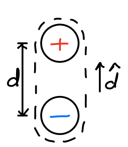
$V_a > V_b$

$$\longrightarrow V = \left\{\frac{L}{\sigma S}\right\}\cdot I$$

$$\text{resistance} = R = \frac{-\int_a^b \vec{E}\cdot d\vec{L}}{\sigma\iint_S \vec{E}\cdot d\vec{S}}$$

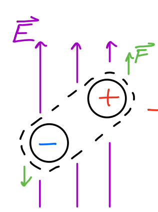
Dielectric

- Dipole: Two charges that are equal & opposite and close together

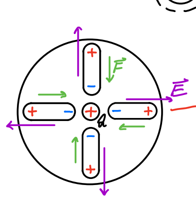


$$\vec{p} = Q\vec{d}$$

electric dipole moment vector



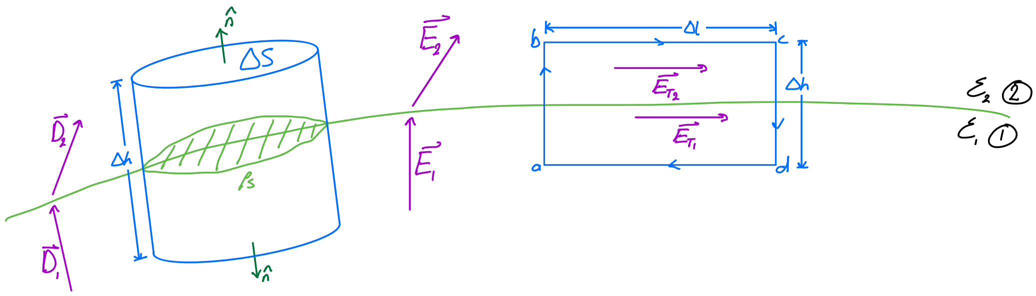
Dipole system rotates without translation



Internal polarization creates opposing $\vec{E}$ fields, reducing the original external field

Electric Boundary Conditions

- Electric field will refract as it passes between dielectrics with different permittivities (ε)



Electric Field: $\lim_{\substack{ab,cd\to 0\\ \Delta h\to 0}}\left(\oint_L \vec{E}\cdot d\vec{L}\right)=0$

$$E_{T_2}\cdot\Delta l - E_{T_1}\cdot\Delta l = 0$$
$$E_{T_2} = E_{T_1}$$

Flux Density: $\lim_{\Delta h\to 0}\left(\oint_L \vec{E}\cdot d\vec{L}\right)=Q_{enc}$

$$D_{N_2}\Delta S - D_{N_1}\Delta S = \rho_s\Delta S$$
$$D_{N_2} - D_{N_1} = \rho_s$$

- In a perfect dielectric (no free charge: $\rho_s = 0$) $\longrightarrow D_{N_2} = D_{N_1} \longrightarrow \varepsilon_1 E_{N_1} = \varepsilon_2 E_{N_2}$
- If one material is a conductor $\rightarrow D_{Nt} = 0$; $D_N = \rho_s$

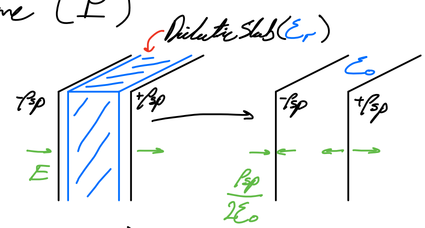
Polarization

- Number of dipole moments per unit volume ($\vec{P}$)

$$\vec{P} = \frac{\vec{p}}{v} = \frac{Q\vec{d}}{Sd} = \rho_{sp}\hat{d}$$

electric dipole moment vector

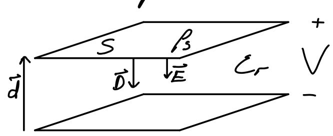
surface charge due to polarization



- Model a dielectric as air

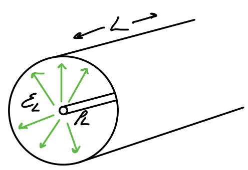
$$\vec{D} = \varepsilon_0\varepsilon_r\vec{E}$$

Capacitance



$$C = \frac{|Q|}{|V|} = \frac{\iint_S \rho_s\, ds}{\int_L \vec{E}\cdot dL} = \frac{\varepsilon\cdot S}{d} = \frac{\varepsilon_r\cdot\varepsilon_0\cdot S}{d}$$

Ex:



$$C = \frac{\rho_L L}{V}\ ;\ V = \int_{ring}^{center}\vec{E}\cdot d\vec{\rho} = \frac{\rho_L \ln\left(\frac{ring}{center}\right)}{2\pi\varepsilon_L\varepsilon_0} \longrightarrow C = \frac{2\pi\varepsilon_L\varepsilon_0 L}{\ln\left(\frac{ring}{center}\right)}$$

- Two stacked dielectrics within plates behave like series, side-by-side behaves like parallel

Magnetostatics

- Biot-Savart Law
  - Relationship between current distribution and the field it generates

current filament

$$d\vec{H} = \frac{I\,d\vec{L} \times \hat{r}}{4\pi R^2}$$

direction from filament to point

distance between filament to point

$\vec{B}$: Magnetic Flux Density [T]

$\vec{H}$: Magnetic Field Intensity [A/m]

μ_0: Magnetic Permeability

4πE-7

$$H = \oint \frac{I\,d\vec{L} \times \hat{r}}{4\pi R^2} = \oint \frac{I\,dL \sin\theta}{4\pi R^2}$$

angle between $\hat{r}$ & $d\vec{L}$

- Ampere's Circuital Law
  - Line integral of H about a closed path is equal to direct current enclosed by said path

$$\oint \vec{H} \cdot d\vec{L} = I \longrightarrow \vec{\nabla} \times \vec{H} = \vec{J}$$

current density at point of magnetic field

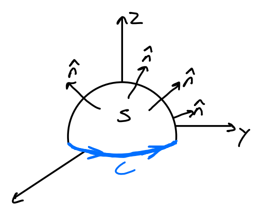
Magnetic Flux [Wb]

$$\Phi = \iint_S \vec{B} \cdot d\vec{s} \longrightarrow \oint \vec{B} \cdot d\vec{s} = 0 \longrightarrow \vec{\nabla} \cdot \vec{B} = 0; \ (\vec{B} = \mu \vec{H}) \left[\frac{Wb}{m^2}\right]$$

$\mu_r \cdot \mu_0$

Stoke's Theorem

- Let "S" be a smooth surface that is bounded by a simple curve "C", then:



$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \vec{\nabla} \times \vec{F} \cdot d\vec{S}$$

Continuity of Current

$$I = -\frac{d}{dt} Q_{enc} = \oiint \vec{J} \cdot d\vec{S} = -\frac{d}{dt} \iiint_V \rho_v \, dV = \iiint_V \vec{\nabla} \cdot \vec{J} \, dV$$

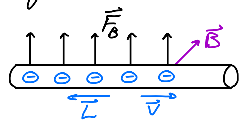
$$\hookrightarrow \vec{\nabla} \cdot \vec{J} = -\frac{d}{dt} \rho_v$$

- Describes how charge enters/exits a system

Magnetic Forces/Torques

- Steady magnetic force is produced by moving charges → current

$$\vec{F}_B = Q\vec{v}\times\vec{B} = i\,\vec{L}\times\vec{B}$$



- Torque on Current loop
  - Current (I) encloses area (S) with normal ($\hat{n}$);
  - Magnetic Dipole Moment: $\vec{p}_m = NIS\cdot\hat{n}$

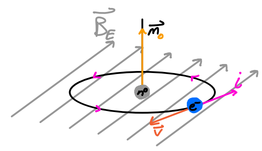
$$\vec{\tau}_B = \vec{p}_m \times \vec{B}$$

N: # of coil turns; I: current

$$d\vec{\tau}_B = I\,d\vec{S}\times\vec{B}$$

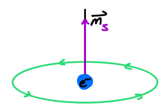
Magnetic Materials

- Electrons orbiting the Nucleus creates magnetic dipoles ($\vec{m}_o$)
- External field lines ($\vec{B}_E$) produce torque ($\vec{\tau}_o$) on the system



$$\vec{\tau}_o = \vec{m}_o \times \vec{B}_E$$

- Electron spin generates magnetic dipole moments
  - Simplified and approximated value of 9.28E-24 [A/m^2]



- Diamagnetic
  - Magnetic moments produced by the spin and the orbit of the electrons cancel out: $\vec{m}_s + \vec{m}_o = 0$ → external fields ($\vec{B}_E$) do not produce torque
  - External fields are repelled by diamagnets → $\vec{B}_{in} < \vec{B}_{out}$
- Paramagnetic
  - $\vec{m}_s + \vec{m}_o \doteq 0 \longrightarrow \vec{B}_E \doteq 0$
  - When an external field is applied paramagnetic dipoles align to enhance external fields → $\vec{B}_{in} > \vec{B}_{out}$
- Ferromagnetic
  - Large atomic moments caused by greater electron spin → $\vec{m}_s \gg \vec{m}_o$
  - Domain: regions of atomic dipoles that are all magnetically aligned
  - Overall moment of ferromagnet is 0, unless it has been exposed to an external field before
    - When exposed to $\vec{B}_E$; some domains align increasing $\vec{B}_{in}$ to be much larger than $\vec{B}_{out}$
  - Hysteresis: newly realigned dipoles do not return to original alignment in absence of $\vec{B}_E$
  - Curie Temperature: above this temperature (570 °C) feromagnets revert to paramagnets
  - Soft Feromagnets: easy to magnetize/demagnetize
  - Hard Feromagnets: hard to magnetize/demagnetize

Magnetization

- Magnetic Dipole Moment due to single orbiting electron:

$$\vec{p}_m = i_B\,d\vec{S}$$

i_B: bound current; $d\vec{S}$: surface area of electron's path

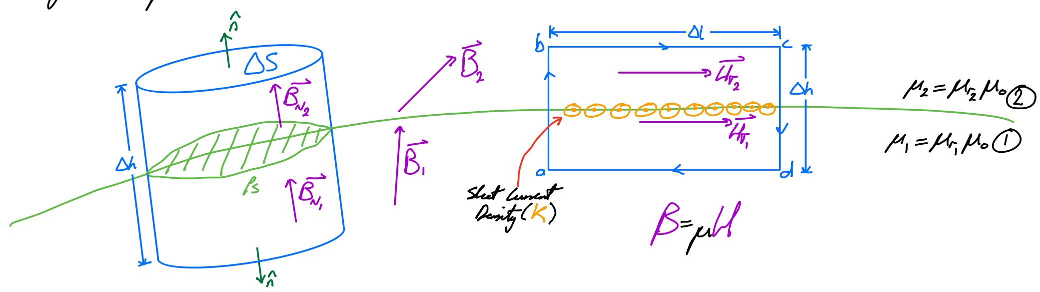
- Sum of Dipoles (n) in volume (ΔV) is $\vec{M}$

$$\vec{M} = \lim_{\Delta V\to 0}\frac{1}{\Delta V}\sum_{i=1}^{n\Delta V}\vec{p}_{m_i} \longrightarrow i_B = \oint \vec{M}\cdot d\vec{L}$$

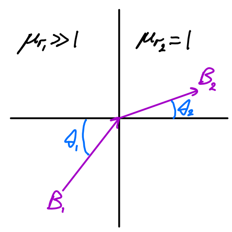
- Total current (i_T) through closed loop: $i_T = \oint \left(\frac{\vec{B}}{\mu_0} - \vec{M}\right)\cdot d\vec{L} \longrightarrow \vec{B} = \mu_0(\vec{H} + \vec{M})$
- "$\vec{M}$" is zero in free space
- Magnetic Susceptibility (χ_m): $\mu_r = 1 + \chi_m$; $\vec{M} = \chi_m \vec{H} \longrightarrow \vec{B} = \mu_r \mu_0 \vec{H}$

Magnetic Boundary Conditions

- Magnetic field



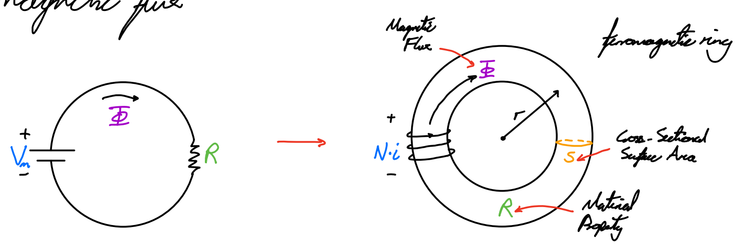
- Normal component of magnetic field is continuous: $B_{N_1} = B_{N_2}$
- Tangential component: $H_{T_1} - H_{T_2} = K$
  - $K = 0 \longrightarrow \mu_{r_2} B_{T_1} = \mu_{r_1} B_{T_2}$
- If material has much higher permeability than medium, the incidence field angle approaches 0



$B_{T_2} = \frac{B_{T_1}}{\mu_{r_1}}$; as $\mu_{r_1} \to \infty$, $\theta_2 \to 0$

Magnetic Circuits

- A magnetic circuit is made up of one or more closed loop paths containing magnetic flux



- Magnetomotive Force ($V_m = Ni$) [Amp-Turns]
  - Wrap current carrying coil around magnetically permeable material
  - Insert permanent magnet into ring
- Reluctance ($\mathcal{R}$) [1/H]
  - Flow of flux around magnetic circuit is resisted by reluctance of magnetic circuit

$$\mathcal{R} = \frac{L}{\mu S}$$

length of material

Magnetic Permeability

  - Flux flows through path of least reluctance

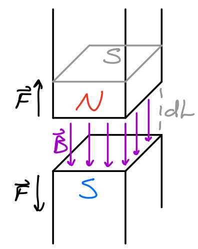
- Magnetic Flux (Φ) [Wb]
- Ohm's Law applied to magnetostatics: $V_m = \Phi \cdot R$

Magnetic Potential Energy

$$W_H = \frac{1}{2}\iiint_V \vec{B}\cdot\vec{H}\,dV = \frac{1}{2}\iiint_V \frac{\vec{B}^2}{\mu}\,dV$$

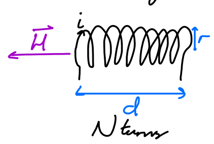
- Applies to magnetically linear mediums such as air
- Magnetically opposed poles have an attractive force between them
- Consider the work done to separate the poles:

$$dW_H = F\,dL = \frac{B^2 S\,dL}{2\mu_0} \longrightarrow F = \frac{B^2 S}{2\mu_0}$$



Solenoids

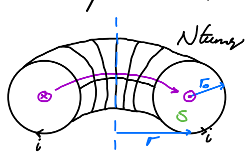
- Coil of wire used as an electromagnet
- Concentrates the magnetic field down center of coil



$$H = \frac{N \cdot i}{d}\ [A/m]$$

Toroids

- Donut-shaped coil of wire used as an electromagnet



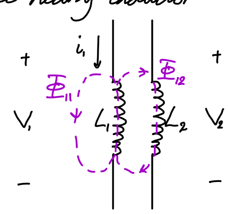
$$H = \frac{N \cdot i}{2\pi r}\ [A/m]$$

Inductors

- Device for storing energy in a magnetic field
- Generates magnetic flux lines which form closed loops
  - These lines may also pass through other coils

- Self-Inductance is the ratio of total flux linkage to the linked current: $L = \frac{N\Phi}{i}$
- Mutual Inductance
  - Change in magnetic flux may result from nearby coil
  - The ability of one inductor to induce a voltage across a nearby inductor

$$\Phi_1 = \Phi_{11} + \Phi_{12}$$



$$V_1 = N_1 \frac{d\Phi_1}{dt} \qquad V_2 = N_2 \frac{d\Phi_{12}}{dt_1} = M_{21}\frac{di_1}{dt}$$

$$M_{21} = N_2 \frac{d\Phi_{12}}{dt_1} = N_2 \frac{\Phi_{12}}{i_1}\ [H]$$

- Physical Inductance: $L = \mu \dfrac{N^2 S}{l}$ — S: cross section of inductor; l: length of inductor
- Energy: $W_H = \dfrac{Li^2}{2}$
- Energy Conservation bounds Mutual-Inductance

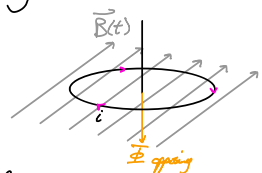
$$M \le \sqrt{L_1 L_2} \longrightarrow K = \frac{M}{\sqrt{L_1 L_2}} = \frac{\Phi_{12}}{\Phi_{11} + \Phi_{12}} \;;\quad 0 \le K \le 1$$

0: No magnetic flux crossover; 1: Maximum magnetic flux crossover

  - K is a measure of how much magnetic flux passes from one coil to the other

Induced Electromotive Forces (Faraday's Law)

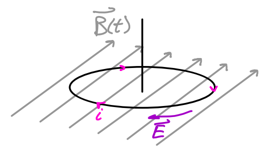
$$emf = -N\frac{d\Phi}{dt}\;[\text{Volts}] = -N\frac{d}{dt}\iint_S \vec{B}(t)\cdot d\vec{S}$$



- emf can be produced by:
  1) Change in $B(t)$
  2) Change in surface that flux is passing through
  3) Change in B with location

$$emf = \oint_L \vec{E}\cdot d\vec{L} \neq 0$$

Not zero in time-varying case



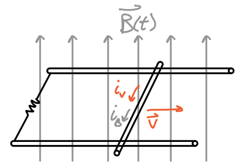
- stationary loop in time-varying B field results in:

$$\vec{\nabla}\times\vec{E} = -\frac{d\vec{B}(t)}{dt}$$

- moving conductor in time-varying B field results in:

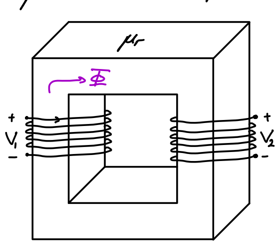
$$\vec{\nabla}\times\vec{E} = \vec{\nabla}\times\vec{E} + \vec{\nabla}\times\vec{E}$$

$$= -\frac{d\vec{B}(t)}{dt} + \vec{\nabla}\times(\vec{v}\times\vec{B})$$



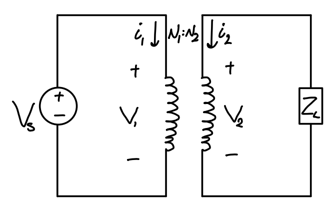
Transformer

- Step-Up transformer: $V_2 > V_1$; $i_2 < i_1$
- Step-Down transformer: $V_2 < V_1$; $i_2 > i_1$



$$V_1 = N_1\frac{d\Phi}{dt}$$

$$V_2 = N_2\frac{d\Phi}{dt}$$

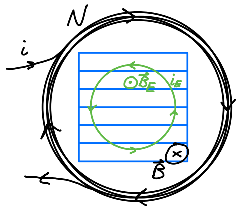


$$\frac{V_2}{V_1} = \frac{N_2}{N_1}$$

$$\frac{i_2}{i_1} = -\frac{N_1}{N_2}$$

- At steady state, the emf and Φ in core are same as before load was connected

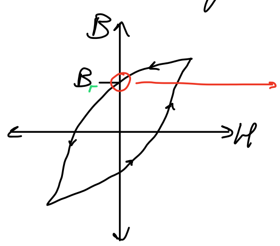
- Eddy Currents
  - Loops of current induced within conductors by a changing $\vec{B}$ field



$\vec{B}_E$ is induced by $\vec{B} \rightarrow$ generates Power loss: $P_L = i_E^2 \cdot R$

  - Sheets of ferromagnets that are insulated from each other prevent Eddy Currents

- Hysteresis: Magnetic Flux Density lags behind Magnetic Field Intensity
  - Newly realigned dipoles do not all return to their previous arbitrary directions
  - Hysteresis Loop: The relationship between B & H as material is magnetized and demagnetized repeatedly



even when the magnetic field intensity (H) is 0, there is still residual flux density (B_r)

  - Hysteresis Loss: Every cycle of the Loop results in energy lost to heat due to the friction generated from dipole movement

Displacement Current (I_d)

- Displacement current density ($\vec{J}_d$) is the result of time-varying electric flux density $\left(\frac{d\vec{D}}{dt}\right)$
- Ampere's Circuital Law becomes:

$$\vec{\nabla} \times \vec{H} = \vec{J} + \vec{J}_d$$

$$I_d = \iint_S \frac{d\vec{D}}{dt} \cdot d\vec{S} \qquad\qquad \oint \vec{H} \cdot d\vec{L} = I + I_d$$

Maxwell's Equations

$$\vec{\nabla} \times \vec{E} = -\frac{\delta \vec{B}}{\delta t} \qquad \vec{\nabla} \times \vec{H} = \vec{J} + \frac{d\vec{D}}{dt} \qquad \vec{\nabla} \cdot \vec{D} = \rho_v \qquad \vec{\nabla} \cdot \vec{B} = 0$$